DEPARTMENT OF ECONOMICS

THE DEVELOPMENT OF GAME THEORY

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(TRANSLATED BY ERIKA SCHWÖDIAUER)

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This paper is a translation by Erika Schwödiauer of the paper "The Development of Game Theory", by Professor N.N. Vorob'ev, which he appended to the Russian translation of Theory of Games and Economic Behavior, by John von Neumann and Oskar Morgenstern; the translation was published in Moscow by "Science" in 1970, in an edition of 14,000 copies.

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THE DEVELOPMENT OF GAME THEORY

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New York University
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A day will come when, thanks to studies extending over centuries, things which today obviously appear obscure to us will only cause our descendants' surprise that such evident truths had escaped us.

-SENECA

INTRODUCTION

Usually a "genealogical tree" is conceived of as a tree in the sense of graph theory whose branches spring from only one "root". The genealogical tree of game theory reminds one rather of the concept of a tree in its original botanical meaning. It possesses numerous ramified roots originating in the remoteness of past centuries and developing into a trunk - the book by J. von Neumann and O. Morgenstern - and a mighty crown interlaced with the contemporary contributions to the theory of games. The tree is just about to bear fruit, the practical crop being yet to come.

Therefore, in the development of game theory, from a mathematized to a mathematical discipline, there is a natural classification of the historical progress into three periods. The first period - until the appearance of J. von Neumann's and O. Morgenstern's monograph - may be called "pre-monographic". At this stage a game appears as a concrete contest described by its rules given in non-mathematical terms. Only at the end of this period did J. von Neumann elaborate the concept of a game as a general model of an abstract conflict. The outcome of this epoch has been the accumulation of a series of concrete mathematical results and even of certain principles of the future theory of games.

The second period is constituted by J.von Neumann's and O. Morgenstern's monograph itself which incorporated most of the results obtained earlier (which were, by the way, not too numerous according to present mathematical standards). It has offered for the first time a mathematical approach to games (both in the concrete and in the abstract sense of this word) in the form of a systematic theory. In the history of mathematics only few books can be found which, like J.von Neumann's and O. Morgenstern's monograph, have established a complicated, important, and at the same time unconventional mathematical discipline practically in the "empty space".

In its third stage game theory differs eventually only little in its approach to the objects to be studied from other mathematical disciplines; and develops at a considerable rate in a way similar to all of them. Besides, it is clear that the specific characteristics of its actual, as well as potential, applications have a decisive impact on the development of the different branches of game theory.

What has been said also determines the general structure of this survey article. Its chapters correspond to the historical periods of game theory outlined above. Obviously, the present article cannot pretend to be an exhaustive account of all facts which are important for the history of game theory. That should be left to special investigations.

C h a p t e r   I

BEFORE THE MONOGRAPH

§1: The Indeterminateness of the Outcome of a Game and Its Sources

1. Since games of the competitive type are models or imitations of conflicts, an indeterminateness of outcome is usually characteristic. This circumstance especially prompts those contestants who are doomed to defeat from the beginning, to enter consciously into the conflict. And it is this same circumstance that attracts both the participants in a competition and its bystanders. Eventually, owing to this fact, each decision made by a player in the course of a game turns out to be subject to indeterminateness.

Quantitative characteristics of indeterminateness as studied by the theory of information (or the theory of complexities) undoubtedly influence decision-making under such conditions. In the framework of game theory, however, these influences have been insufficiently explored and only some episodical studies of game theoretic character have been dedicated to this problem.

From a purely qualitative point of view the causes of indeterminateness in the result of a game can be divided into three classes.

2. Let us begin with the case where the rules of the game allow for such a multitude of plays that an *a priori* prediction of the outcome of each play is practically impossible, while, in principle, abstracting from the

difference between potential and actual feasibility, no obstacle to such a prediction exists.

Sources of indeterminateness of this type can be called combinatorial, and games whose outcomes are unpredictable for these combinatorial reasons can be called combinatorial games. Chess, for example, is a typical combinatorial game.

Obviously, the combinatorial complexity of a game is of a historically transitory character. The development of certain methods of playing a game "correctly", sometimes generalized in the form of a suitable mathematical apparatus, makes most variants of playing the game more and more transparent, and the utilization of computing techniques is extending the concept of "transparence" itself.

At present, several games are in different stages of this historical process. Those games for which this process has come to an end practically lose their competitive character and become merely entertaining; they may still be, however, of some pedagogical and sometimes also scientific value. That has happened, for instance, to the well-known "Nim" games where the players alternately take objects from some baskets according to the given rules. These games can be completely formalized, and finding the winning combinations (if such winning combinations exist) reduces to the solution of not too voluminous logical problems.

Although the purely logical principles do not allow an extensive analysis of other more complicated games, they often lead to some predictions of general character. Eventually, in games of a complexity comparable roughly to that of chess, the logical considerations do not go beyond the sphere of common sense (which may be sometimes nevertheless rather acute), and the aptitude of analyzing, evaluating and comparing large numbers of variants becomes a central part of the art of playing a game.

3. The second source of indeterminateness in the outcome of a game is the influence of chance factors. Chance can appear in a game either as the result of certain "natural forces" (dispersion of shots, meteorological conditions, random causes of overflows in channels of mass service systems, etc.) perform "randomized" acts organized in a special manner (tossing coins or throwing dice, using tables of random digits, etc.).

Games whose outcomes are indeterminate solely by virtue of chance are called games of hazard. Typical examples of games of hazard are games of dice of any variety and also the game "Matching Pennies" in the special form where one player tosses a coin and his adversary tries to guess the side it shows. A game of pure hazard is also the well-known roulecte. There is nothing to be said about the correct or optimal behavior of a player in a game of hazard: The outcome of the game does not depend on his actions. The only decisions the player is able to make concern the advisability of his participation in or his absence from a game depending on its rules. Such decisions belong however to a considerable degree to the psychological shpere (see I.3.4 *)).

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*) Here and further, references to parts of this article are denoted as above. In references to J. von Neumann's and O. Morgenstern's monograph the work "Chapter", the symbol "§", and the letter "i", are used in front of the respective numbers of chapters, paragraphs, and items.

Of course, games can be found which combine features of both combinatorial games and games of hazard. To this class of games belong, for example, Backgammon (Tric-Trac) with its presently rather numerous variants and also various kinds of card solitaires where the indeterminateness is created on the one hand by the chance arrangement of cards in the card deck, and on the other hand by the combinatorial complexity of the configurations made up by the uncovered cards on the table.

4. The third source of indeterminateness in the outcome of a game is of strategic origin: The player may not know what action his adversary chooses. Contrary to the two previously mentioned sources of indeterminateness, this one is founded in the very nature of a game. It originates from the other participants in the game which may be real (man, collectivity) as well as conditional (nature, circumstances). Games in which the indeterminateness of the outcome stems from the indicated strategical causes are called strategic games. Strange as it may seem strategic games in their pure form are relatively rare. The simplest example of a strategic game is the game "Matching Pennies" in the form where two players independently of each other put a coin on the table. If both coins show the same side the first player wins, otherwise the second. Despite all its primitivity this game appears to be in some respect "more difficult" than, say, chess. A play of chess takes place on an open board, and it is possible to imagine an "ideal player" who overlooks all possibilities emerging from each position. Each move thought through by one of the players is deliberated to the same extent by his opponent. In contrast to that, in the described game "Matching Pennies"

a player is unable in principle to recognize what his adversary has done. It is especially this feature that makes such a game a strategic one.

As to the correctness and optimality of the behavior of a player, matters here are essentially more complicated than in the previously described cases. It is obvious that putting the coin heads or tails cannnot per se mean good or bad behavior, for as N. Wiener has remarked,"...the efficacy of a weapon depends on precisely what other weapons there are to meet it..." /1/.

Actually, in strategic games optimal behavior is randomized. Applied to "Matching Pennies" this means that it is not advisable to put the coin on the table showing a specific side, but to toss it in such a manner that each of its sides can fall with equal chances.

The strategic property of a game can be combined with a combinatorial property ("kriegsspiel" - a kind of chess where each player playing on his own board sees only his figures and an umpire removes them in case of their capture, announces checks, and decides on checkmate or stalemate), or a hazard property (poker), or with both combinatorial and hazard properties at the same time ("Preference" - where the hazard property comes from the chance deal of cards, the strategic property results from the prescriptions of the game and the determination of "pairing", and the combinatorial property comes from the difficulty to orientate oneself in the cards even if they are uncovered).

§2: Combinatorial Games

1. Evidently, combinatorial games in the form of mathematical problems made their first appearance at the beginning of the seventeenth century. The well-known "Problèmes Plaisants et Délectables, qui ses font par les Nombres", published by Bachet de Méziriac in 1612 /1/, contains a problem of the following sort: Two players alternately name numbers from one to ten, and the player who arrives first at 100 with the numbers added up wins.

The solution of this game causes no troubles: A player can be sure of winning if he succeeds in making the sum of all numbers named equal to 100 - 11a. Accordingly, after each step of his opponent, he has to choose a number being a complement to 11. As a matter of fact, by naming number one at his first step, the beginning player can force his conquest. This game can obviously be interpreted as a process of alternately taking away one to ten objects from a basket originally containing 100 objects.

The Chinese game "Fan-Tan" seems to be more complicated. In this game, the players have to cope with three baskets filled with certain objects, and at each move the respective player has to choose an arbitrary number of objects from an arbitrary basket. The player who takes the last object wins. The complete theory of this game has been published in 1902 by C.L. Bouton /1/.

This type of game in its more general form is called "Nim", and it is of the following structure: There are given n baskets with certain objects. Each of both players, who are alternately coming in, chooses p baskets and from each of these he takes an arbitrary number of objects. The player taking the last object wins. An analysis of this game has been given by E.H.Moore in 1909 /1/.

Moore's arguments utilize in fact the following consideration: It can be shown that for every game of the type "Nim" it is possible to obtain in an obvious way some class of positions possessing the so-called properties of external and internal stability. The first property means that whatever the position not belonging to the class investigated is, there exists a step leading from it to a position in our class. The second property consists in fact that each move made in a position of this class leads outside. In this way, if the class described contains the winning position (and in games of "Nim" the player who takes the last objects wins), each position in this class can be regarded as winning.

In particular, in Bachet de Méziriac's game those positions in which it was possible to represent the sum of chosen numbers in the form 100 - 11a turned out to be the class of winning positions.

The concept of dual stability introduced above has proved extremely fruitful in game theory. We will come back to this concept repeatedly later on.

2. With the increasing combinatorial complexity of a game, plainly determining the set of all winning positions (in the sense described above, i.e., possessing the property of dual stability) becomes more and more difficult. For a game like chess this turns out to be practically impossible. Therefore, in the mathematical analysis of combinatorially complicated games, efforts shift from the search for the set of winning positions to proving the existence of such sets.

This way has been chosen by E.Zermelo. In 1912, at the Fifth International Congress of Mathematicians, he presented his paper "Über eine Anwendung der Mengenlehre auf die Theorie des Schachspiels" /1/, in which he proposed the following approach to conbinatorial games.

Consider a game with a finite set of positions (Zermelo, for the sake of security, speaks only of chess, but he has all similar games in extensive form in mind). Positions that differ only with respect to which player's turn it is to move are regarded as different.

For each position q we introduce a set $U_r(q)$ of such "end games" where White can force his victory in not more than r moves. Here the possibility of forcing a victory is to be understood in the following sense.

Let some end game $\zeta = (q, q_1, q_2, \dots)$ belong to the set $U_r(q)$, q_λ being some position in ζ where it is Black's turn to move, and he moves from q_λ to the position $q_{\lambda+1}$. Let us consider another position, $q'_{\lambda+1}$, which, according to the rules, can also be reached by Black from q_λ. Then among the end games in $U_r(q)$ an end game ζ' can be found that starts with the positions $q, q_1, \dots, q_\lambda, q'_{\lambda-1}$. The possibility for White to force his victory from q in r steps means that $U_r(q) \neq \emptyset$.

If the total number of all possible positions in the game equals t, then from White's possibility in position q to force his conquest in a finite number of moves follows an analogous possibility of victory in not more than t moves. Thus White's possibility to win in position q is equivalent to $U(q) = U_t(q) \neq \emptyset$.

In a similar way, the set $V(q) \supset U(q)$ of end games beginning in q, where White is able to force a tie, is determined. A tie is attainable for White if $V(q) \neq \emptyset$. If $V(q) = \emptyset$, then in position q Black forces the victory.

By means of this argument Zermelo showed that in each position q one of three possibilities exists: Either White can guarantee himself the victory (if $U(q) \neq \emptyset$), or he can force a tie but not a victory (i.e., Black can guarantee himself a tie but not the victory, this happens, when $U(q) = \emptyset$, but $V(q) \neq \emptyset$), or the victory can be forced by Black (if $V(q) = \emptyset$). The same question may also be asked for the starting position. As Zermelo has pointed out, an answer to this question would deprive chess completely of its game character.

3. In 1925 Steinhaus published his article "A Definition for a Theory of Games and Pursuit" which for years had been known only to a small circle until it appeared in 1960 in an English translation (under the above title). In this article Steinhaus introduced (for the sake of security again applied to chess) the concept of a "method of playing" as a "list of all eventualities with a preferable move for each of them". The best strategy is regarded as that method which minimizes the maximum number of steps one's adversary can persevere. In fact, the ideas of a strategy and a maximin-principle are already contained in these definitions.

In his paper, Steinhaus examined neither the questions of existence nor of finding the best strategies referring them to another class of problems (according to the general hierarchy of problems given at the beginning of his paper).

4. In Zermelo's considerations, the transition from forcing the victory in a finite number of steps to forcing the victory in a bounded number of moves has not been duly proven. In 1927, D. König /1/ has given a precise proof of this proposition on the basis of one of his theorems from the theory of infinite graphs. The possibility of applying this theorem to games has been pointed out to him by J. von Neumann. It was apparently in that paper that J. von eumann's name was mentioned for the first time in connection with games. By the way, J. von Neumann had already shown his interest in problems of games earlier. More details about that will be given in I.4.1.

For the sake of fairness it must be noted that Zermelo, when he became acquainted with König's paper before its publication, offered his own independent, extremely brief and elegant version of the lacking proof described by König in the appendix to his article /1/. According to König, the proof based on this idea had been known also to J. von Neumann.

5. All in all, König's proof does not rely on the condition of finiteness of the number of possible positons in the game, but only on the weaker condition of finiteness of the number of positions attainable in one move from a given position and thereby in n moves for an arbitrarily fixed natural n. As an example for such a game he has used a game played with ordinary chessmen on an unbounded board.

The consequent step in this direction has been made by L. Kalmár who dropped the condition of finiteness of the number of positions attainable from a given position in one move.

6. The formalization of the argumentation referring to the dual stability of winning sets in two-person games in extensive form with alternating moves has been carried out by P. Grundy /1,2/. A pair of positions in a game that differ only with respect to which player's turn it is to move is called a diagram. Let us denote the set of all possible diagrams in a game by X. If we represent each diagram by a point and connect it by directed arcs with all those diagrams which can be reached in one step from the given one, then we obtain an orientated graph, denoted by (Γ, X). Let us attach the payoff of a player to the end diagrams of this graph (i.e., those diagrams from which one cannot pass to other diagrams).

Let each player win (we limit ourselves to the description of a case that may be called symmetric) in diagrams of the set K at his move, and in diagrams of the set L at his opponent's move. This determines a game denoted by (Γ, X, K, L).

The function g, defined on the set of all diagrams whose values are non-negative integers, is called Grundy function if it possesses the following properties:

$$g(x) = \begin{cases} 0 \text{ for } x \in L, \\ 1 \text{ for } x \in K, \\ \text{the smallest non-negative integer *)} \\ \text{different from } g(y) \text{ where } y \in \Gamma x. \end{cases}$$

Let there exist a Grundy function for a given game (Γ, X, K, L). For this, as M. Richardson /1,2/ has subsequently shown, it is sufficient that

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*) Translator's note: The author speaks here of "natural number", but from the following argumentation obviously follows that he really means "non-negative integer".

in the graph (Γ, X) both the set Γx and the set of all $y \in X$, for which $x \in \Gamma y$, are finite, for any $x \in X$, and likewise that this graph does not contain contours of odd length.

The set of those diagrams where the Grundy function takes on the value zero possesses the property of dual stability.

Let us then assume that some player (say, the first) succeeds in attaining by one of his moves a diagram a, for which $g(a) = 0$. Now the second player chooses a diagram $b \in \Gamma a$. But, according to the definition of a Grundy function, the value $g(a)$ being equal to zero differs from all numbers $g(z)$ for $z \in \Gamma a$. In particular, $g(b) \neq 0$ must hold.

We see that it is not possible to perform a transition within the set of zero-Grundy function *) in one step. This property of the set is called its internal stability.

Let now the first player be in position b such that $g(b) \neq 0$. He chooses some $c \in \Gamma b$. If among the diagrams Γb no diagram z, for which, $g(z) = 0$ could be found, then $g(b) = 0$ would follow, which, however, does not hold. Consequently, there exists a $c \in \Gamma b$ such that $g(c) = 0$.

This means that it is always possible to pass from outside the set of zero-Grundy function into this set by one step. This property of the set is called its external stability.

Thus, the first player once being in the set of zero-Grundy function has the possibility not to leave this set. Thereby he either carries the game into the set L or prevents it from terminating.

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*) I.e., the set of diagrams on which the Grundy function takes on the value zero (translator's note).

§3: Games of Hazard

1. The chance factor is the prominent one in all games of hazard (it may be helpful to remember that in French "hazard" means "chance event" and stems from the Arabic word "azar" - az-zahr signifying "difficult"; originally this expression had been used to characterize the rarest event).

Above all games of dice can be reckoned as hazard games. For a long time several variants of these games had been the main source of problems in the theory of probability and its only field of application. We observe that throwing dice had not only been practised for competitive purposes, but also for soothsaying where every combination of eyes had its own meaning.

Let us turn to the following two circumstances. In an overwhelming majority of hazard games (accordingly also in games of dice) chance does not emerge as a spontaneous action of some "elemental" forces, but as the result of conscious acts of people participatins in the game. Moveover, the use of dice for soothsaying means that randomizing arrangements had been accepted for the resolution of problems of decision-making.

2. The first discussion of probability and to some extent even calculations concerning various outcomes of throwing dice can apparently be found in Cardano's treatise "On Hazard Games" (see for instance H.G. Zeuthen /1/, p. 168), and an exhaustive analysis of the probabilities of various outcomes of casting three dice is contained in Galilei's study "On the Outcome of Eyes in Games of Dice" (see the article by Maistrov /2/).

Then, only half a century later, there is the well-known exchange of letters between Pascal and Fermat, where (in Pascal's letter to Fermat from

from July 29, 1654) the solution to de Méré's problem was discussed. It is regarded, according to a general tradition, as the origin of the mathematicel theory of probability. About this time (1657) Huygens finished his treatuse "On Calcuation in Games of Hazard" /1/, where he writes among other things: "...when studying the subject carefully the reader will observe that he is dealing not only with a game, but there are given the fundamentals of a deep and extremely interesting theory." Only fourteen years later, however, Jean deVitte applied the probability calculus to the calculation of the values of life interests, and from then on probability theory as a branch of mathematics has left its association with games and has begun an independent existence. As to the role of hazard games in the rise of probability theory se Maistrov's paper /1/.

3. If the player's aim in a combinatorial game is winning it and optimal actions or strategies of the player are regarded to be those which assure him this win, then under conditions of a hazard game no skill (that does not transgress the rules of the game) can guarantee the player the desired outcome which depends besides other factors also on chance. For that reason, the player cannot by merely choosing a strategy, obtain a fixed amount. Here the purpose seems to be much more complicated.

The tendency to maximize the payoff which he expects to receive seems most natural for the player. The quantitative evaluation of the hopes of the players in several games (actually under conditions of an unfinished match consisting of various plays) had already been subject of a controversy between Cardano and Lucca Paccioli in the sixteenth century (see Zeuthen /1/, p.168), and a century later in Pascal's above mentioned letter it had been taken as a basis for a "fair" division of stakes not payed out. Huygens arrived independently of Pascal and Fermat at a similar result expressed in a much more general form which allows us to speak of the mathematical expectation.

Thus the maximization of the mathematical expectation of the payoff turned out to be the leading principle for the participant in a game of hazard. Later on, Laplace /1/ included this principle in his collection of "fundamental principles of the calculus of probabilities" (principle VIII) and formulated it as follows: "If an advantage depends on many events, then by taking the sum of the products of the probabilities of each event that is randomly connected with its occurrence one obtains this advantage." Laplace declared that this advantage means the mathematical expectation. On these grounds the idea of a harmless game as a game where the mathematical expectation of the gains of every player is zero at the beginning has emerged.

If one chooses to deal with games of hazard from the point of view of maximizing the mathematical expectation of gains, they can in principle be exhaustively analyzed by means of probability theory. The difficulties one may meet will be of a purely technical character. We will therefore not dwell upon further mathematical discussions of hazard games based on this principle.

4. An uncritical application of the principle of maximization of the mathematical expectation may lead to paradoxical results. The first example of such a paradox has been given by Nikolaus Bernoulli and has been named "St. Petersburg Paradox". It is of the following kind.

Two players flip a coin till "tails" turns up. If "tails" appears for the first time at the n^{th} throw the first player receives from the second 2^n units. The mathematical expectation of the payoff for the first player is infinite. Therefore, whatever is the amount of his original stake, the game will never be "harmless" but advantageous for him. This conclusion is, however, not in accord with "common sense", for the second player's capital is practically limited and even through adjourned plays the first player will not be able to get his due gain. Besides, the first player's "capability of appropriation" is also limited. Therefore, at sufficiently large n a gain of 2^n with probability $(1/2)^n$ is preferable to a gain of 2^{n+1} with probability $(1/2)^{n+1}$: Both gains are "practically equally huge", but the probability of the first is greater than that of the second.

The two objections quoted are essentially different. The first is more formal and can be just as formally abandoned if one identifies potential and actual realizability.

The second objection is notwithstanding its apparent deliberations more substantial: It reflects the circumstance that an increase in utility arising from an increase in the monetary gain does not only depend on this increase, but also on the absolute size of the gain.

Daniel Bernoulli (Nikolaus Bernoulli's nephew) assumed that the utility of an increase dx in the gain were directly proportional to dx and indirectly proportional to x. As it can easily be seen this is equivalent to the statement that the utility of the monetary gain be proportional to its logarithm. From this follows that gaining a certain amount of utility and losing it afterwards is just as profitable for the player as losing and then regaining it, for in both cases he loses a smaller part of his capital than he obtains. This statement is also to some extent paradoxical. In any case, it is possible to find evident arguments in favor of and against it.

Measuring utility on such a logarithmic scale leads to the replacement of the mathematical expectation by the "moral expectation" which should be better called "psychological expectation". Such a measurement of utility is also included by Laplace in his principles of probability calculus (principle X), who qualifies it, however, only as a principle "that might be useful in many cases".

5. The theory of hazard games has been developed in its most general and perfect form by L. Dubins and L. Savage in their monograph /1/. They have formulated the fundamental problem of the theory as that one of finding the optimal behavior of a player, who, at the beginning of a game, can dispose of a certain sum and possesses a given utility function.

The most essential point is that classical probability distributions (i.e., countably additive probabilities) have turned out to be inadequate for a complete description of phenomena arising in games of hazard. For that reason Dubins and Savage have found it necessary to develop a more general theory - finitely additive discrete stochastic processes.

In the following we will see that finitely additive probability distributions are important for games of strategy too.

§4: Games of Strategy. E. Borel's contribution

1. In contrast to both the combinatorial and chance aspects of games whose mathematical development roots in the remoteness of centuries, the strategic problems of games have a considerably shorter history.

The first mathematical discussion of the strategic aspect of games can be found in J. Bertrand's lectures on probability theory /1/ where the question about the advisability of "drawing for five" in Baccarat had been investigated. Bertrand's reasoning is to a considerable degree psychological in character: He evaluates the expediency for a punter to draw in dependence on the banker's knowledge about the punter's usual behavior.

What Bertrand has said is not to be valued as properly raising a mathematical problem but only as pointing out its possibility.

2. In 1921 a short, but absolutely substantial note by E. Borel on "The Theory of Play and Integral Equations With Skew-symmetric Kernels" /1/ appeared. In this paper fundamental concepts concerning games of strategy (strictly speaking, those games which later on have been termed symmetric antagonistic games) have been formulated for the first time.

A strategy has been defined as a system of rules determining exactly the actions of a plyaer in any possible situation. Here the game has been looked upon as a hazard game, and the choice of strategies c_i and c_j by the two players A and B, respectively, leads to player A's win with probability $1/2 + \alpha_{ij}$ (from the symmetry of the game follows that $\alpha_{ij} = -\alpha_{ji}$ and $\alpha_{ii} = 0$).

Borel has also expressed the idea of utilizing the domination of strategies: If $\alpha_{ih} \leqq 0$ for all values of h, then strategy c_i can be called "bad" and excluded from further considerations.

On the contrary, if for all values of h $\alpha_{ih} \geqq 0$ holds, then c_i can be reckoned as the "better" strategy.

In case where bad strategies have been abandoned and better strategies are lacking one has to attempt to develop a system of play based on a alternation of strategies which does not depend on psychological considerations but on the rules of the game. Borel has considered the unique possibility to be the choice of strategy c_k by player A with a certain probability p_k. Analogously, player B will select his strategy c_k with probability q_k. Thus, Borel for the first time stated the expediency of using mixed strategies.

The choice of such mixed strategies results in a probability of winning for player A of

$$\sum_i \sum_j (1/2 + \alpha_{ij})\ p_i q_i = 1/2 + \alpha$$

It is not difficult to see that is the number of each player's strategies is three, then

$$\alpha = \begin{vmatrix} p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \\ \alpha_{23} & \alpha_{31} & \alpha_{12} \end{vmatrix} .$$

It is further clear that in case of the absence of "bad" and "better" strategies the numbers α_{23}, α_{31}, and α_{12} have equal signs, and that it is possible to find a system of probabilities p_1, p_2, p_3, for which α will be zero independent of what the system of probabilities q_1, q_2, q_3 is going to be.

Thus, using current language, Borel has proved the theorem of the existence of optimal strategies (also called minimax-theorem) for symmetric matrix games of dimension 3 x 3.

Obviously, this proof depends on rather concrete deliberations having to do with the fact that the number of pure, non-randomized strategies for each player equals three. Borel has been in doubt about the possibility of extending this result to the case of an arbitrary number of strategies, and has even been inclined to give a negative answer for the general case. He conjectured that "in general, whatever the p's may be, it will be possible to choose the q's in such a manner that α has any sign determined in advance.

However, when making this conjecture Borel has employed an argument that shows his deep understanding of strategic situations leading to mixed strategies. He writes: "Since this is the situation, whatever variety is introduced by A into his play, once this variety is defined, it will be enough for B to know it in order that he may vary his play in such a manner as to have an advantage over A. The reciprocal is also true, whence we should conclude that the calculation of probabilities can serve only to facilitate elimination of bad manners of playing and the calculation of α_{ij}; for the rest, the art of play depends on psychology and not on mathematics."

Only now are we able to truly appreciate the reasoning. We will come back to this question later on.

Besides, Borel has extended the problem to the case of continuous strategy sets of the players. Here discrete probabilities are changing to probability distributions and sums to Stieltjes integrals. As an example he takes a curious problem which is to some extent a prototype of the contemporary problems of distribution of infinitely divisable resources: Each of the players A and B chooses three numbers adding up to unity,

$$x + y + z = 1,$$

$$x_1 + y_1 + z_1 = 1,$$

and orders them in a certain manner; A wins if

$$(x_1 - x)\ (y_1 - y)\ (z_1 - z) > 0$$

(for another interpretation of this game see II.5.3).

At the end of his paper Borel points out that the probabilistic and analytical problems that may arise in the art of military strategy, or in economics and financial affairs, are not without similarity to the game problems examined.

Thus, in this first paper dedicated to games of strategy, Borel has raised rather than solved the problem, but he has done it in an entirely profound way, even from the present point of view.

3. In a further paper /2/ published in 1924 Borel returned to the examination of strategic games. In this article he has given an exhaustive analysis of symmetric matrix games with three and five strategies for each player. Incidentally, in one of the footnotes Borel has pointed out the possibility of symmetrization of an arbitrary game based on the players' participation in two plays of such a game and their performance of different roles in these plays (a precise description of this symmetrization has been given by Brown and von Neumann 1950 /1/).

We notice, at first glance, another somewhat strange phenomenon similar to that we have already observed in the seventeenth century when Pascal and Fermat had in fact employed the formula for absolute probabilities, i.e., the mathematical expectation of conditional probabilities, but not the mathematical expectation of the gain as such. The latter had been given by Huygens. Similarly, Borel in his 1921 note has not dealt with numerical gains, but with averaging probabilities of winning. The mathematical expectation of the gain itself can be found only in his 1924 paper.

This can be partly explained by the fact that the formula for absolute probabilities replaces certain probabilities by another probability. The mathematical expectation of gains, however, although measured in the same units as the gains themselves, is essentially not of the same nature. In order to be able to interpret it as a gain, additional stipulations that are not always evident and even not always natural are necessary. Full clarity about this problem has been achieved only after establishing an axiomatic theory of utility, which will be touched upon in II.

The remainder of Borel's 1924 paper as well as his further publications on this topic do not contain anything new compared to his note of 1921.

4. In 1953 English translations of Borel's contributions /1, 2, 3, 4/ appeared together with a short preface by M. Frechet /1/ entitled "Emile Borel, initiator of the theory of psychological games and its applications", in Econometrica. As a true mathematician, Frechet gives the definition of the term "initiator", a term he uses, quoting Legouve: "I call initiators those privileged beings, those magnetic creatures who make vibrate in us cords until then mute, those wakers of soul."

But just in this sense Borel cannot be considered an initiator of the theory of "psychological" (strategic) games. His papers on game theory in the 'twenties have wakened nobody's soul, nor have they found any responses in publications. The title of an initiator of game theory could be legitimately awarded to Borel even if he had wakened the soul of "only" a J. von Neumann.

But as can be seen from J. von Neumann's footnote to his paper /1/, he had learned of Borel's note only while his paper had been put into its final form.

Borel's name does not need any titles. But to characterize his place in the history of game theory in one word it seems most appropriate to use the term "discoverer", who came, saw, and ... that's all. Borel has seen a lot, however.

§5: On the Theory of Games of Strategy

1. In 1926 J. von Neumann dealt with problems concerning games*). He discussed them with D. Konig, and on December 7, on the eve of his twenty-third birthday he presented a report on game theory to the Gottingen Mathematical Society. One and a half years later von Neumann's paper "Zur Theorie der Gesellschaftsspiele" has been published in the 100th volume of the "Mathematische Annalen". This paper contains the most important ideas of the present theory of games and its fundamental results.

Von Neumann himself has formulated the purpose of his article as an attempt to give an answer to the following question: " n players S_1, S_2, ..., S_n are playing a given game of strategy, G. How must one of the participants, S_m, play in order to achieve a most advantageous result?". Of course, he has not mentioned all possible practical interpretations of this question, but he has noted that "there is hardly a situation in daily life into which this problem does not enter". Yet, the meaning of this question is not unambiguous in J. von Neumann's opinion.

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*) See the note on p. 13 in volume 171 of collected paper.

In fact, the principal achievement of this paper is the clear mathematical formulation of this question that has not only allowed us to answer it, but also to show the specific "calculus" for this kind of question.

The paper begins with the definition of a game of strategy with n players $S_1, \ldots, S_n$ as a specification of a set of events ("steps", "moves") carried out either by the players or by chance. Thereby the game G is described as a zero-sum extensive game (i.e., for an arbitrary outcome of the game the sum of payoffs to all players equals zero). Such games are also called antagonistic games.

Von Neumann has further found out that actually the strategies of the players are systems of possible actions in several states of information. This makes it possible to restrict oneself, in solving fundamental theoretical problems, to the normal form of the game where the strategies of a player are examined without taking into account their origin, i.e. as elements of an abstract set of strategies.

Cases where the number n of participants in a game equals 0 or 1 are not of much interest. The case of $n = 2$ is not only the simplest of the non-trivial cases, but, as it will turn out in the following, is of principal importance for the entire theory. Hence, von Neumann proceeded to a detailed investigation of two-person zero-sum games whose rules he has formulated as follows:

"The players S_1, S_2 choose arbitrary numbers among the numbers 1, 2, ... Σ_1, and 1, 2, ... Σ_2 respectively, each one without knowing the choice of the other. After having chosen the numbers x and y respectively, they receive the sume $g(x,y)$ and $-g(x,y)$ respectively. $g(x,y)$ may be any function (defined for $x = 1, 2, \ldots \Sigma_1$; $y = 1, 2, \ldots \Sigma_2$)."

After this there follow arguments which in our time seem like cliches or even primitive, but it is them which constitute the core of game theory separating it from other branches of mathematics. There is, for example, the argument that player S_1 for an arbitrary choice x receives a payoff not less that $\min_y g(x,y)$ and has therefore to choose his x such as to maximize this minimum, i.e., he guarantees himself a sum of $\max_x \min_y g(x,y)$.

Player S_2 need not give S_1 an amount greater that

$$\min_y \max_x g(x,y).$$

The equality

$$\max_x \min_y g(x,y) = \min_y \max_x g(x,y)$$

frees the optimality of the players' actions of any psychological touch. Those values of x and y in this equality where the respective outer extrema are attained, obviously are the optimal strategies of players S_1 and S_2. The parts of this equality can generally be interpreted as the sum player S_1 is sure to obtain, but which is also the most he can win if his opponent plays correctly. This sum is called the value of the game.

Von Neumann has attempted to overcome the difficulty that this equality is not always fulfilled by the same method as Borel a few years before him, i.e., by introducing mixed strategies $\zeta = \zeta_1, \ldots, \zeta_{\Sigma 1})$ and $\eta = (\eta_1, \ldots, \eta_{\Sigma 1})$. But Borel has expressed the logical aim in the form of a rather complicated assertation (see I.4.2). Von Neumann has added to this the equality

$$\max_{\zeta} \min_{\eta} h(\zeta,\eta) = \min_{\eta} \max_{\zeta} h(\zeta,\eta) \qquad (1)$$

where $h(\zeta,\eta)$ is a bilinear form:

$$h(\zeta,\eta) = \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} g(p,q)\, \zeta_p\, \eta_q \, .$$

Thus von Neumann has established the existence of a value and optimal (possibly mixed) strategies for the players of an arbitrary finite antagonistic game. Thereby, he has dispelled Borel's doubts and refuted the conjecture to which Borel's scepticism tended.

The proof of the equality (1), given by von Neumann, is by all means complicated and not constructive. It is based on Brouwer's fixed point theorem. This is the more surprising as the proposition to be proved had of course been put sometimes before in terms of convex sets (Minkowski /1/) and linear inequalities (Stiemke /1/). However, ten years had yet to pass until Ville /1/ revealed the connection between this game problem and the theory of convex sets and has given an elementary proof of the minimax equality.

2. After the principal basis of the behavior of participants in an antagonistic game (i.e., a two-person zero-sum game) appeared to be perfectly clear, von Neumann went on to the analysis of games with more than 2 participants. But already, in the case of three-person games (and even for two-person nón-zero-sum games), additional difficulties arose. Not trying to overcome them (incidentally we note that until the present day, no one has succeeded in establishing an exhaustive theory of three-person games),

von Nuemann chose to investigate the possible ways of reduction of many-person games to antagonistic games. He has examined in detail these reductions for the case $n = 3$ and has outlined an analogous program for games with more than three players. Von Neumann's idea consists in the following:

Let us consider a three-person zero-sum game where the players S_1, S_2, and S_3 choosing independently of each other strategies $x = 1, \ldots, \Sigma_1$, $y = 1, \ldots, \Sigma_2$, and $z = 1, \ldots, \Sigma_3$ respectively, receive the payoffs $g_1(x,y,z)$, $g_2(x,y,z)$, and $g_3(x,y,z)$ where identically

$$g_1 + g_2 + g_3 \equiv 0.$$

We consider all possible antagonistic games obtained as the result of an association of any two arbitrary players against the third (obviously, in the given case there are three such antagonistic games). We determine the values of these games:

$$\max_{\zeta} \min_{\eta} \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} [g_1(p_1q_1r) + g_2(p_1q_1r)] \zeta_{pq} \eta_r = M_{1,2}$$

$$\max_{\zeta} \min_{\eta} \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} [g_1(p_1q_1r) + g_3(p_1q_1r)] \zeta_{pr} \eta_q = M_{1,3}$$

$$\max_{\zeta} \min_{\eta} \sum_{p=1}^{\Sigma_1} \sum_{q=1}^{\Sigma_2} \sum_{r=1}^{\Sigma_3} [g_2(p_1q_1r) + g_3(p_1q_1r)] \zeta_{qr} \eta_p = M_{2,3}$$

Here, the ζ_{pq} form a system of probabilities on the set of pairs (p,q); analogous distributions are given by ζ_{pr} and ζ_{qr}. It is not difficult to show that

$$M_{1,2} + M_{1,3} + M_{2,3} \geq 0 \qquad (2)$$

The goal of each player is to receive the largest possible payoff. What claim, say of player S_1, may be considered realizable? Let us assume that S_1 makes efforts to obtain w_1. His adversaries united with each other give him not more than $-M_{2,3}$. That means his claim is realizable if

$$w_1 \leq -M_{2,3} \ . \qquad (3)$$

Otherwise it would be necessary for player S_1 to form a coalition with one of his partners in the game.

If S_1 having in view to gain w_1, gets S_2 as an ally, then there remains $M_{1,2} - w_1$ for S_2, and if S_1 is in coalition with S_3, then S_3 obtains $M_{1,3} - w_1$. This means that S_2 and S_3 "together" receive $M_{1,2} + M_{1,3} - 2w_1$. But, by rejecting S_1's offer to coalesce and forming a coalition among themselves they can get $M_{2,3}$. If

$$M_{2,3} > M_{1,2} + M_{1,3} - 2w \ ,$$

then there is both for S_2 and S_3 no sense in responding to S_1's call who thus will be left in isolation. Consequently, in order to find an ally, S_1 has to moderate his claim:

$$w_1 \leq 1/2\,(M_{1,2} + M_{1,3} - M_{2,3}) = \bar{w}_1$$

(which in view of (2) is less restrictive than (3)). Analogously, we find that the payoffs w_2 and w_3 desired by players S_2 and S_3 respectively, are correspondingly restricted by the inequalities

$$w_2 \leq 1/2\,(M_{1,2} + M_{2,3} - M_{1,3}) = \bar{w}_2 \ ,$$
$$w_3 \leq 1/2\,(M_{1,3} + M_{2,3} - M_{1,2}) = \bar{w}_3 \ .$$

But at the same time the claims $\bar{w}_1$, $\bar{w}_2$, and $\bar{w}_3$ are realizable for any pair of players: A coalition made up from a pair of players is able to satisfy the demands of its members. Of course, the player remaining outside the coalition thereby "robs" himself completely.

3. For an arbitrary number n of players, similar arguments can be put forth. In particular, the arbitrary splitting of the whole set of players into two coalitions $\mu_1, \ldots, \mu_k$ and $\nu_1, \ldots, \nu_{n-k}$ opposing each other, determines for any coalition $\mu_1, \ldots, \mu_k$ an antagonistic game and its value $M_{\{\mu_1,\ldots,\mu_k\}}$ As can easily be seen,

1) $M_{\{\}} = 0$;

2) $M_{\{\mu_1,\ldots,\mu_k\}} + M_{\{\nu_1,\ldots,\nu_{n-k}\}} = 0$

if the set-theoretical sum of the coalitions $\mu_1, \ldots, \mu_k$ and $\nu_1, \ldots, \nu_{n-k}$ makes up the entire set of players:

3) $M_{\{\mu_1,\ldots,\mu_k\}} + M_{\{\nu_1,\ldots,\nu_e\}} \leq M_{\{\mu_1,\ldots,\mu_k,\nu_1,\ldots\nu_e\}}$

if the coalitions $\mu_1, \ldots, \mu_k$ and $\nu_1, \ldots, \nu_e$ are disjoint subsets.

The properties of the magnitude M as a function of coalitions are at the same time the essential properties of the original game. Later on this function has been given the name <u>characteristic function</u> and has been examined in many papers.

The approach described reminds us to some extent of the theory of correlation of random variables, where the study of joint distributions of many random variables is confined to finding and comparing the correlation coefficients of any possible pair of random variables. In the role it plays in the theory of games, the characteristic function introduced above reminds us of the correlation function. Of course, this similarity is of a purely logical, but not formal nature.

4. We see that von Neumann's paper contains a lot of fundamental ideas of the current theory of games of strategy, and the history of game theory begins with it. Hence, it is justifiable to call von Neumann the founder of game theory. Yet, during the immediately following years, von Neumann's article found neither responses nor continuations in the contemporary mathematical literature. The sole exception is the already mentioned paper by Ville /1/ containing the simplified proof of the minimax-theorem and its extension to games with infinite sets of strategies obtained on the basis of an inquiry into games of the Poker type. In particular, in that paper /1/, Ville proved that any infinite antagonistic game, where the set of strategies of each player is a unit segment (these games are now called games on the unit square) and the payoff function is continuous, has a value in mixed strategies. By the way, the example of a game (with an infinite set of strategies, of course,) having no value (in mixed strategies) has already been given in this paper. Thus the question formerly raised by Borel (see I.4.2), concerning the possible limitations of the probability approach to games of strategy and the necessity of a psychological approach, has arisen again.

Consequently, when speaking about the 'twenties and 'thirties, von Neumann himself cannot be regarded the "initiator" of game theory (in the Legouvé-Frechet terminology).

It may be noted that practice (especially in planning of experiments) has revealed, obviously independently of von Neumann's game theoretical ideas, the advisability of employing mixed strategies. Analyzing the game "Le Her", R. A. Fisher /1/ has pointed out the rationality of a random choice of one of two procedures for each player and has even given a detailed calculation of the respective probabilities.

5. At the end of the 'thirties von Neumann occupied himself anew with problems of game theory, but this time together with the economist O. Morgenstern. As a result of their inquiry into the theory of games, they have determined, if not the immediate direction of applications, then in any case, a kind of "social order": Game theory has become ripe for its elaboration as a mathematical apparatus for the description and analysis of economic phenomena. The fruit of a work of many years has been the monograph Theory of Games and Economic Behavior.

Chapter II

THEORY OF GAMES AND ECONOMIC BEHAVIOR

Our intention is not to give a brief exposition of the contents of the monograph or any of its passages. There is also no need for "literary critical" considerations concerning what, in particular, the authors had in view when organizing their scientific material in that and no other way, nor in what inspired the specific composition of the book and its sections. The authors themselves have described, and they have done this in considerable detail, every logical step in their mathematical reasoning and have discussed thoroughly the necessity to expose the problems in the order eventually adopted.

At the same time, it seems to be advisable to make some remarks of game theoretical as well as generally methodological character. Besides, it appears natural to point to those results obtained later which immediately refer to concrete problems analyzed in the monograph.

For convenien all these remarks are arranged in the same order as the passages of the monograph they are referring to. They are grouped into paragraphs that correspond to the chapters of the monograph.

§1: Formulation of the Economic Problem

1. The book is called "Theory of Games and Economic Behavior" and its authors begin by pointing out their purpose - "to present a discussion of some fundamental questions of economic theory". Thus the impression may arise that this book is dedicated to economics and particularly to the applications of game theoretic methods in economics. Really, however, the contents of the monograph are purely mathematical.

The authors have realized that the formally described results of mathematical conclusions may, despite the perfect strictness of their exposition, appear by no means indisputable from the point of view of those readers who are not familiar with the mathematical apparatus used. Therefore, everywhere the possibility occurred they have accompanied the mathematical formulations by purely verbal descriptions.

The economic inclination of the book manifests itself in only three aspects that are not decisive, but nevertheless deserve some explanation.

First, the starting point of the authors' inquiry has been the discussion of some elementary economic phenomena like isolated exchange acts, aspects of competition, equilibrium etc. Here they confine themselves to a detached consideration of those phenomena, ignoring their general economic and social features, which thus lose the specific quality that transforms them from immediately observed facts into economic phenomena. From the point of view of economic science, this is inadmissable; but mathematically - completely legitimate, since it facilitates the construction of a formal theory taking into account only those features of phenomena that are preserved after a certain schematization has been carried out.

Secondly, an economic terminology has not infrequently been employed in the book: There occur terms like money, monopoly, duopoly, bilateral monopoly, etc. As to monopolies, duopolies etc., they are not to be understood as concepts from bourgeois political economy, and even not as terms pointing to some phenomena which are in fact or at least apparently observable in the capitalist economy. In the book these terms denote only certain variants of collision of opposing interests. To what extent these variants correspond to actual collisions of economic interests under capitalistic conditions has not been investigated by the authors of the book.

Somewhat more complicated is the question of money. By money the authors mean, as they put it on page 8, "a single monetary commodity" supposed to be "unrestrictedly divisible and substitutable, freely transferable and identical", even in the quantitative sense, with whatever "satisfaction" or "utility" is desired by each participant.

It is clear that this axiomatically defined commodity does not possess all properties of money. It is also obvious that money does not always, or completely, possess the properties postulated by the definition. At the same time, this commodity is much more "similar" to money than to something else, and, for that reason, calling it money is more convenient than using any other name. Nevertheless, the term "money" is used by the authors only in concrete illustrative examples. In all their theoretical discussions they prefer the term "utility".

Thirdly, the authors' eventual goal, which goes beyond the scope of their book, is indeed the economic applications of the mathematical apparatus developed by them. But of what nature these applications will actually turn out to be and to what economic conclusions they will lead, the authors have not shown anywhere in the book and have not intended to do so.

As a matter of fact, the theory of games has various economic (as well as other) applications. In this paper, however, we choose not to deal with problems of application.

2. The notion of utility, as a quantitatively measurable and unrestictedly divisible object, is one of the most important notions in game theory. The existence of such a utility could be postulated from the very beginning as an axiom which is then subjected to scrutiny in every single case. However, on the one hand, the assertion of the existence of a utility with the properties mentioned is not sufficiently unquestionable. Therefore, von Neumann and Morgenstern prefer to base the requirements that utility must meet on several more elementary axioms.

Here we may point to the peculiar axiomatic treatment of most of the fundamental concepts in game theory. Usually, in formal mathematical theories, axioms are chosen not so much for their naturalness, simplicity, and originality as for deductive reasons: independence, consistency, possible completeness, transparency, etc. Roughly speaking, this accounts for the small number of axioms and the large number of theorems in traditional axiomatic theories.

In game theory, matters are essentially different. Statements about the existence of necessary objects (including, anticipating something, also principles of rational behavior) presupposed "axiomatically" do not always look sufficiently probable (the more so not always absolutely probable) and are therefore not accepted by everybody. For that reason, the problem arises of formulating "more primitive", "more probable" axioms, and on the basis of them proving certain game theoretic principles originally regarded as elementary. Not infrequently, for the sake of proving a single theorem of which virtually the whole theory consists, an extensive axiom system is worked out.

Precisely such an axiom system has been given by the authors in 1.3.6, and the existence of the required "utility function" has been derived from it in the Appendix (pp. 616 - 630).

3. The theory of games has been developed by von Neumann and Morgenstern as a theory of mathematical models of conflicts. Already, in the simpler cases, the role of the information a player has about the behavior of his partners comes to light. So, in the case of an antagonistic game with a payoff function $H(x,y)$, the maximizing player knowing his opponent's choice of strategy (for instance, by virtue of the fact that he makes his choice after him) receives with certainty $\min_y \max_x H(x,y)$, and in the opposite case, i.e., not knowing anything about his adversary, he can only count on $\max_x \min_y H(x,y)$. At the same time, from a mathematical point of view, it makes no difference at all whether this adversary is a real agent acting consciously and, besides that, harmfully, to our player or a fictitious one personifying only the player's insufficient knowledge about the situation in which he has to make his decisions. Such opponents may be, for example, nature whose regularities may be perceived insufficiently at the moment of decision-making, or, say, a person completely well-intentioned towards the player but guided in his actions by criteria unknown to him.

Thus the theory of games can be viewed also as a mathematical apparatus describing decision-making under conditions of uncertainty which can be naturally termed strategic uncertainty. Such an approach to the theory of games (in particular to antagonistic games) has been systematically put forth by A. Wald in his book Statistical Decision Functions /2/. All in all, most of the military tactical applications of game theory (see for instance M. Dresher's book /1/ and also the collection of papers /7/) are based not so much on the hostile intentions of the enemy (we will return to this question in III.3.3) as on the unpredictability of his actions. The technical applications of game theory are based on these considerations, examples of which are given by N.N.Vorob'ev in /7/.

Strategic uncertainty, which game theory deals with, differs fundamentally from statistical uncertainty. Statistical uncertainty prevails in those cases where the decision-maker does not know the true state of affairs but knows the a priori probabilities of all possible variants of situations. In case of strategic uncertainty, the agent does not have any reasons for ascribing a priori probabilities to these possible variants.

Correspondingly, under conditions of strategic uncertainty, one has to introduce and employ a concept of information different from that used for statistical uncertainty. For the case of statistical uncertainty, a theory of "selective information", which is now sufficiently well-known, has been developed by C. Shannon /1/ and his followers. For the game-theoretical strategic uncertainty, the notion of "strategic information" introduced by M. Sakaguchi /1/ is of great importance.

4. Von Neumann and Morgenstern have often illustrated their discussions of the paths of development of game theory and its applications by facts taken from the history of thermodynamics. Hence, they speak in 1.3.2. about the possibility of obtaining a rigid numerical scale for temperature based on the study of the behavior of ideal gases, and about the role of absolute temperature in connection with the entropy theorem.

Therefore, we may note that the numerical scale for temperature had been developed long before the nature of ideal gases and of other physical conceptions mentioned was clarified. Its construction was based on the immediately observed phenomenon of expansion of heated bodies. At the time, this phenomenon had not been logically connected with the phenomena of heat, thus remaining (in any case until the molecular-kinetic theory had been developed) in an external relationship to them. Nevertheless, it has subsequently turned out (in particular from the examination of ideal gases, etc.) that the expansion caused by heat is, fundamentally, purely energetically connected with the measurement of temperature, so that the scale constructed has also been the only possible one (up to linear transformations).

Returning to utility theory, we see that at the basis of its measurements there is something extrinsically related to utility, in particular, the probability combinations of utilities. The analogy to thermodynamics may let us hope that this probability approach is actually more intrinsically connected with the determination of subjective preferences and that further investigations of subjective preferences will disclose this connection. Of course, the measurement of utility is a qualitatively more complicated procedure than the measurement of temperature, so that all of what has been said above is only a conjecture about some similarity in the tendencies of the development of two completely different theories.

5. The concept of a solution as a set of imputations, introduced in §4, which generalizes the notion of a maximum, reproduces in the last resort the construction leading to the set of zero-Grundy function (see I.2.5). The solutions are sets of imputations possessing the same properties of internal and external stability applied to the relation of domination of imputations, as the sets of possible zero-Grundy functions applied to the relation given by the graph. Therefore, the multitude of solutions of games turns out to be just as much a natural phenomenon as the existence of Grundy functions with different zero-sets on one and the same graph.

As the sets of imputations are continuous and Grundy functions are virutally defined for discrete graphs, the immediate utilization of Grundy functions for finding the solutions of games, or at least for proving the existence of solutions for some classes of games, is hardly possible. It is not impossible, however, that one may succeed in extending some properties of Grundy functions to the continuous case and apply them to game theory. In this connection, the results of M. Richardson /1/ give us some reason for hope.

§2: General Formal Description of Games of Strategy.

1. Von Neumann and Morgenstern, in their monograph, have reproduced and detailed the original definition of a game of strategy introduced by von Neumann in his paper /1/. This definition turns out to be extremely capacious:

Most of the contributions to game theory which have appeare afterwards refer to games just in that sense of the word, maybe somewhat specified or generalized, but all changed only a little.

In particular, these games have subsequently been named non-coalition games and have turned out to be one of the widest (as for its size) classes of games. For an examination of non-coalition games see III.4.

2. However, from a broader point of view, some parts of the given definition seem to be superfluously restricting. Above all, the receipt of the individual payoff has been considered the goal of every participant in a game by von Neumann and Morgenstern. Even in those cases (examined by the cooperative variant of the theory) where the players form a coalition for joint actions, they do this in order to share among themselves some total payoff obtained for the entire coalition.

At the same time, in the economic and social reality, it may happen that the payoff obtained by a coalition belongs to this coalition as such, and is not subjected to a further division among the players composing it. In particular, it can turn out that one and the same player is simultaneously a member of two or more different coalitions whose interests do not coincide. It is clear, that game theory laying claim to the sufficiently complete analysis of conflicting interests of various parties, has to reflect this aspect of the problem too. To this question we will return again in III.4.7.

Further, in connection with the treatment of game theory as a mathematical theory of decision-making under the conditions of uncertainty, a critical attitude comes up towards one of the fundamental theses of game theory requiring that the players are completely informed about the conditions (rules) of the game they are participating in.

From an intuitive point of view the player's knowledge about a game means the fulfilment of two conditions: 1) each player knows the aim he is striving for and 2) each player is perfectly aware of the consequences which are implied by the selection of a certain strategy.

Formally, however, there is in principle no difference between these conditions, and the mathematical treatment of games where the first or the second of these conditions (or both) are not fulfilled can be carried out by one and the same scheme. It is natural to call such a game indeterminate. Rudiments of the theory of indeterminate games are contained in N.N.Vorob'ev's paper /5/.

Eventually, von Neumann and Morgenstern have assumed the finiteness of the set of players in every game. While in the future, games with infinite sets of players will be without doubt studied more intensively than games with finite sets of players, up to the present day, only papers by Shapley /5/, Davis /1/ and also Kalisch and Nering /1/ have been dedicated to games with infinite sets of players.

3. As in von Neumann's paper /1/, the general definition of a game is given in extensive form. This not only corresponds to the actual course of play in most games (games in a literal sense of the word as well as conflicts or processes of decision-making modelled by games) but also reflects the fact that the player in the course of play makes his decisions on the basis of the information he has and which can change in the course of play. In particular,

the player's information about his own past states of information and about those decisions he made in them can change. This last circumstance is described in the monograph in terms of "preliminarity" and "anteriority". In the following these discussions have served as a starting point for studies undertaken by Kuhn /1/ and subsequent papers to which we will come in III.5.2.

4. The inclusion of chance moves into a game allows for an examination of games being simultaneously games of strategy and of hazard. The presentation of the set of positions in the form of an orientated graph shows that the general definition of a game of strategy also comprehends the combinatorial aspect.

The fact that the authors have confined themsleves to the case of a discrete set of moves, is, from a game-theoretical point of view, not particularly essential, though, of course, the transition to a continuous set of moves like, for instance, in differential games (see III.5.7), involves considerable difficulties.

5. Strategies are introduced by von Neumann and Morgenstern in the course of "final simplification" of the description of a game, resulting in the definition of a game in the normalized, i.e., in the purely strategic form. While such a simplified description is actually equivalent to the original, apparently, however, it represents the description of a more particular object: Games where each player makes only one move in complete ignorance about the moves made by each of the other players. That has motivated the authors to speak in the following (1.12.1.1) of an extensive game as the extensive form of the game.

At the same time extensive games are acutally more concrete objects than games in the normalized form. In fact, the fundamental concept for games of strategy is the concept of a strategy. In games in the normalized form, the strategies of the players are deprived of substantial properties whatever those may be, being simply elements of some abstract set. In extensive games, the strategies appear as functions on the set of all information states of the player, i.e., as objects of an essentially more concrete nature, endowed with individual properties.

What has been said above determines also the expediency of utilizing the description of games in the normalized form in some cases and in the extensive form in other cases. As the authors have remarked in 1.12.1.1, the examination of games in the normalized form is better suited for the derivation of general theorems being relevant for whole classes of games, for formulations of general principles of optimal behavior of the players etc. Games in the extensive form are preferably used for the analysis of peculiarities of the behavior of the players in a given game, for determining possible reductions of strategies etc. We note at the same time that in obtaining the actual <u>solution</u> of games, i.e., in finding optimal (or in another sense, expedient) strategies of players in some concrete games, one has been successful up to now only for games in the normalized form with the exception of some very specific cases.

§3. Zero-Sum Two-Person Games: Theory.

1. In 1.12.2 von Neumann and Morgenstern have examined games with only one player. From the mathematical point of view, finding the rational behavior of the participant in such a game consists in solving a maximization problem and is not of game-theoretical interest.

2. Zero-sum two-person games, i.e., antagonistic games, are the simplest ones in a game-theoretical sense. Here the conflicts of two parties appear in an immediately strategic form, and are not complicated by any considerations concerning the entrance of players into a coalition or the exchange of information among them.

In fact, in an antagonistic game, the payoffs to the two players are of equal magnitude and opposite sign. Therefore, if some common action of the players is useful for one of them, i.e., results in an increase of his payoff, then at the same time it decreases the payoff of the second player, i.e., it is not desirable for him. That means that in order for the player to be willing to go with each other on this or that issue, it is necessary that their common action is not of advantage for anyone of them. But in that case, such action will in general not have any effect on the outcome of the game and can be excluded from consideration.

Antagonism in the mathematical, game-theoretical sense, meaning an equality in magnitude and opposition in sign, differs essentially from the philosophical concept of the same name. This one has to bear in mind when one discusses possible ways of modelling conflicts actually encountered by means of games or, vice versa, of empirical interpretations of game-theoretical constructions.

In particular, the sufficiently adequate modelling of social or military conflicts by means of antagonistic games succeeds only in single cases. Actually, in such conflicts each side usually pursues its own aims and inflicting a damage on one's rival is only a method of attaining one's aim or even simply, only an attendant circumstance.

By the way, the antagonisticity of the conflict should not be confused with its keenness. So, for example, the military tactical situation where from each side one unit of armed forces is participating and the aim of each side consists in the annihilation of the opponent's unit, is not antagonistic from the game-theoretical point of view. Under conditions of an antagonistic conflict, the striving for destruction of the opponent is counterbalanced by the striving for escaping one's own destruction.

For a more detailed presentation of this sort of questions see the paper by N.N.Vorob'ev /4/.

3. As the leading principle of optimal behavior for a participant in an antagonistic game, von Neumann and Morgenstern have offered the maximin (minimax) principle. The application of this principle by each of the players results (with employing mixed strategies, if needed) in the value of a game as a "fair" payoff to the first player in that game. The "fairness" of the payoff, equal to the value of the game, can be interpreted as the right of the player to obtain this sum instead of participating in the game. The probability approach to utility allows us to consider the mathematical expectations of payoffs as real payoffs.

The authors have proved the maximin principle after a very detailed analysis of the majorant and the minorant games. These reasonings are essentially of an axiomatic character and can be completely formalized. There is, however, one inconvenience limiting the range of possible applications of the theory. The examination of the majorant and the minorant games involves the simultaneous description of the aims of both players. Therefore, all that has been said above will be sufficiently convincing without any additions and specifications in applications to decision-making under conditions of a conflict between two parties pursuing opposite goals; but in the case of decision-making under conditions of uncertainty, where the decisions are in fact made only by one side, some doubts may remain.

These doubts have been dispelled by E.I.Vilkas /1/ who proposes a partitioning of the maximin principle into a few more particular principles which he uses as axioms.

Let v be a function defined on the set of all matrices. We shall conceive of $v(A)$ as that fair payoff which a person participating as the first player in the matrix game with payoff matrix A can count on. It is natural to require that the function v possesses the following properties.

1° If A and $\tilde{A}$ are two matrices of the same dimension where $A \leq \tilde{A}$ (the inequality holds for each element) then

$$v(A) \leq v(\tilde{A}).$$

In other words, if the player has the choice to participate (as the first player) in the game with matrix A or in the game with matrix $\tilde{A}$ where $A \leq \tilde{A}$, then the participation in the game with matrix $\tilde{A}$ is not less preferable.

2° If the matrix $\tilde{A}$ is obtained from the matrix A by adding to it a new row not exceeding some convex linear combination of the rows of the matrix A then

$$v(A) = v(A).$$

That means that the player is indifferent between participating in the game with matrix A and playing the game with matrix $\tilde{A}$ where he has at first sight more possibilities.

3° If we consider the real number x as a matrix, then

$$v(x) \geq x$$

(i.e., the participation in 1 x 1-game is not less preferable than the immediate obtaining of the payoff).

4° If A^T is the transposed matrix corresponding to A, then

$$v(A) = v(-A^T),$$

i.e., it makes no difference to participate either in a game with the matrix A or in the game with matrix $-A^T$ *).

This system of axioms is complete in the sense described by the following theorem: The function v, satisfying the axioms 1° - 4° is unique and the value v(A) is the value of a matrix game with the payoff matrix A.

As the value of an antagonistic game is just that value of the player's payoff which he obtains following the maximin principle the given system of axioms confirms also this principle itself.

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*) Translator's note: Obviously, axiom 4° is stated here incorrectly. A correct version would be

$$v(A) = -v(-A^T),$$

i.e., it makes no difference to participate either as the first player in a game A, or as the second player in a game $-A^T$.

The quoted system of axioms has dealt with preferable actions of only one acting person. It is applicable therefore, independently of whether his opponent is real (selecting his strategies consciously) or fictitious (generator of the uncertainty). Thereby the fairness of the maximin principle in case of decision-making under conditions of uncertainty has been established.

4. In spite of the fact that the fundamental definition of a game has been given by von Neumann and Morgenstern in the extensive form, when examining games of this kind they have confined themselves to the important but less typical case where the players have perfect information in the game. The result received (in § 15) generalizing the Zermelo theorem (see I.2.2) is very instructive in its naturalness. Essentially, however, game theory is a theory of optimal decision-making under conditions of uncertainty and, at the same time, under conditions of imperfect information; on the other hand, it is characteristic for the theory of games that optimal decisions of the players are shown to be mixed strategies. Therefore, it is absolutely natural that if the player possesses perfect information in the game, i.e., he is acting, fundamentally in a situation which essentially is not a game, then his optimal strategy has to be not a game-theoretical one, but one corresponding to another level of optimal decision-making. Actually, an optimal strategy turns out to be pure in this case.

5. In §16 the proof of the theorem of the existence of optimal strategies in matrix games based on the properties of convex polyhedra is given. The questions of finding optimal strategies in practice will be examined in III.1.2 - III.1.5.

§4. Zero-Sum Two-Person Games: Examples.

1. The content of game theory as a mathematical discipline consists, first, in establishing principles of rational, expedient, optimal behavior of players; secondly, improving the existence of actions of players satisfying these principles and, thirdly, in actually finding such actions. Here the first constitutes the essence of the theory of games as such, the second makes it objective and assures the possibility of applications in principle, and the third turns this possibility into an actual one.

In chapter III. of the monograph, after solving the first two questions for finite antagonistic (i.e., for matrix) games, von Neumann and Morgenstern have passed to the solution of the third question dedicating to it a separate chapter. When the monograph was written, as well as afterwards, one has not succeeded in finding any general methods for solving games, even not for such relatively elementary ones like matrix games. Therefore, except for numerical methods of solution, the practical results for matrix games are limited hitherto to the examination of examples whose number is not yet large.

The first example of a solution of a symmetric 3 x 3 -matrix game has been given by Borel (see I.4.2). Von Neumann and Mrogenstern have given some new examples.

2. The examination of the 2 x 2-game carried out in 1.18.2, has already shown that the attempts to describe its solution by a single formula are not economical and for the transition to games of larger dimensions also hopeless. Therefore, the solution of games belonging to some class is to be understood as an algorithm analyzing the necessary relations between the game parameters and

deriving on the basis of this analysis a computational formula. A matrix game, however, is given by a matrix, i.e., by a comparatively large number of parameters making necessary the analysis of a host of relations, and the required algorithm (if no specific simplifying devices are introduced) is very cumbersome.

Obviously, the difficulties shown are inevitable and could hardly be overcome if one examines each element of the matrix as an independent parameter bearing some information of its own, independently of other parameters. In fact, these elements of the payoff matrix often appear not as original parameters of the game, but are determined on the basis of other, rather limited, initial data. The calculation of the payoff matrix is in this case an intermediate and not absolutely necessary stage. In practice such a "non-matrix way of the solution of matrix games" means abandoning the utilization of the normalized form of a game and possesses all advantages and inadequacies which have been mentioned in II.2.5.

3. The analysis of a literary conflict exposed in i.18.4.4 attracts attention: The behavior of Sherlock Holmes escaping from Professor Moriarty.

The analysis of conflicts by artistic means and the description of the behavior of their participants in accordance with their aims and possibilities has occupied a prominent place in the belles-lettres of all times. As Luce and Raiffa /1/ have justly noted: "In all of man's written record there has been a preoccupation with conflict of interest; possibly only the topics of God, love, and inner struggle have received comparable attention." It may be added to this that the idea of God as a moral category has been invoked to point to certain principles of resolving conflicts, that love in its very essence is a form of common activities of persons endowed with opposite interests, and that inner struggle consists in the conflict of a person with the ignorance of that true aim which he has to strive for. Thereby, the "theme of conflict" turns out to be one of the most widespread ones in the belles-lettres and in art in general.

The theory of games accomplishes the analysis of conflicts by scientific, in particular mathematical, means. Therefore it is of interest to compare the behavior of participants in a conflict reckoned as optimal by the theory of games with the resolution of this conflict by artistic means.

The first attempt of a systematic approach to this question is contained in the paper by N.N.Vorob'ev /8/.

4. The extensive §19 of the monograph is entitled "Poker and Bluffing". The choice especially of Poker as an example for a detailed analysis, can be explained above all by the fact that Poker of all so-called "parlor" games appears to be the most strategic one: Combinatorially the game is extremely simple and the element of hazard, i.e., of chance, is there reduced to the minimum and is amendable to direct calculation. Therefore the game of Poker is played not for a successive accumulation of advantage (as takes place, say, in chess) but immediately for utilities (usually for money).

Being fundamentally an extensive game, Poker possesses its own specific principles of optimal behavior of players. The utilization of optimal mixed strategies with a positive probability of bluffing, belongs to these principles too. The chance bluffing is one of a few examples of mixed strategies intuitively found and systematically used in parlor games.

§5: Zero-Sum Three-Person Games.

1. Beginning with the fifth chapter, von Neumann and Morgenstern have left the purely strategic ground and introduced into the investigation the cooperative aspect of the problem. Here the results of the analysis turn out to be less complete than in the antagonistic case. The solution of an antagonistic game, whose existence for the finite case has been established in chapter III, points out some way of acting which guarantees the player a sure payoff (equal to the value of the game). Within the cooperative theory, one already fails to obtain such a result or a similar one. The player gains the sum pointed out by the theory not by force relying just on his own possibilities, but only conditionally, given a certain behavior of some other participants of the game. Thus the conditions of realizability of claims (as well as their quantitative characteristics) given by the theory, turn out to be necessary but not sufficient.

On the contrary, in a three-person game (essential) all winning (i.e., two-element) coalitions get one and the same payoff, and all losing (one-element) ones suffer one and the same damage. Thus in the three-person game, the cooperative aspect appears in its purest form: The aim of the player is to join a winning coalition. This idea has been elaborated in detail in chapter X dedicated to the simple games. The essential three-person game is simple (and besides that a majority game, see §50).

2. The concepts of the imputation and the solution as a set of imputations turn out to be new and important in principle. Every imputation can virtually be considered a dilemma the players are confronted with: Either to obtain a sum

provided by the imputation or turn to the game for receiving one's payoff in it. The imputation can be regarded "fair" if both of these possibilities are equivalent for the players. In that respect the fair imputation plays a role similar to the value of the antagonistic game: For the player is indifferent between participating in the game (as the first player) and obtaining the value of the game immediately.

3. An imputation can be conceived of as the outcome of a game, proposed by a person who is external in his relationship to the given game. The possible activities of this person are various imputations and his aim is the acceptance of the imputation offered. If there exist several such persons, then their aims are obviously different and by evaluating their goals quantatively we arrive at a game which, because of its relationship to the initial game, may be called a metagame. Its participants will be called metaplayers. A metaplayer may be interpreted as a person bringing before the whole group of the players some project that affects the interests of each player. Roughly speaking, the metaplayer's aim takes shape through his proposition of dominating and non-dominating imputations.

As in the three-person game two imputations cannot simultaneously dominate each other, the antagonistic metagame can under these conditions be constructed very simply: If α and β are imputations selected by the metaplayers I and II respectively, then

$$H(\alpha,\beta) = \begin{cases} 1 & \text{for } \alpha > \beta \\ -1 & \text{for } \alpha < \beta \\ 0 & \text{otherwise} \end{cases}$$

It is not difficult to check if this metagame coincides with the infinite game described by Borel (see I.4.2). Its solution has been found by A.I.Sobol'ev /1/.

§6: Formulation of the General Theory: Zero-Sum n-Person Games.

1. The fundamental concept of the cooperative theory is the concept of the characteristic function defined on the set of all coalitions. The system of relations (25:3:b), (25:3:b), and (25:3:c) expressing the fundamental properties of characteristic functions is virtually a system of axioms describing the natural properties of the possibilities that every coalition has under the least favorable circumstances, in particular, where all players not joining a coalition conjointly stand out against it.

These axioms are in accord with the fundamental ideas of game theory, for the function whose values are the values of the antagonistic games of coalitions against their complements, fulfill these axioms. At the same time, the system of axioms is complete in the sense that there exist no "generally valid" (i.e., true in all interpretations of games) statements not depending on the axioms given. That has been proved in §26 by indicating a method for the construction of a game with an arbitrary given characteristic function.

All in all, the remainder of the monograph deals not with games, but with characteristic functions. Therefore, in the following, it would be possible to replace (except for a few single sections) the term "game" by the term "characteristic function". The authors have not done that because they have limited themselves to the study of those properties of games which manifest themselves in their characteristic functions.

2. The fundamental problem of the cooperative theory is the formalization of the transition from the given possibilities of each coalition, to the individual possibilities of the players. Here the assumption has been used that the utilities obtained by the players and colaitions can be unrestrictedly transferred to other players and coalitions even without any quantitative change. In other words the problem consists in the construction, based on the characteristic function of the game, of such an <u>imputation</u> or such imputations which for given conditions would be in some sense natural and "fair".

The solution of this problem depends, of course, on those axioms of fairness which have been presented.

Von Neumenn and Morgenstern have offered, in fact, a system of three axioms giving a set of imputations V called the <u>solution</u> of the game.

1° For any imputation $(\alpha_1, \ldots, \alpha_n)$

$$\alpha_i \geq v(i)$$

$$\sum_i \alpha_i = 0$$

2° No two imputations in a solution V dominate each other (the axiom of internal stability).

3° Whatever the imputation α not in V is, there exists an imputation β in V which dominates α .

Defining the solution in this way justifies its name from the point of view of naturalness. The purely mathematical theory of solutions is very substantial. For some results obtained in that direction see in III.3.3 - III.3.7.

However, the reduction of the analysis of the game (at least in the form of its characteristic function) to finding and examining its solutions cannot be recognized as exhaustive.

First, except for trivial ("inessential") games the solution has to consist of more than one imputation. This depreciates very much the normative content of the concept of solution, for even a solution found does not show which payoffs the players receive as the result of the game.

Secondly, many games (including already the simplest of essential ones - the zero-sum three-person games) possesses many solutions. Therefore, it would be desirable to complete the axioms quoted by indicating the selection of a certain solution.

Kuhn and Tucker /2/ have quoted from Wolfe's report some remarks made by von Neumann ten years later as chairman of a "round table" discussion in Princeton, February 1, 1955: "Von Neumann pointed out that the enormous variety of solutions which may obtain for n-person games was not surprising in view of the correspondingly enormous variety of observed stable social structures; many differing conventions can endure, existing today for no better reason than that they were yesterday. It is therefore, still of primary importance to settle the general question of the existence of a solution for any n-person game".

Thirdly, at last, the answer to the question mentioned by von Neumann turned out to be negative. Only recently, Lucas /1/ has given an example of a ten-person game not possessing a solution. Thus it turns out that the system of axioms for the solution does not correspond completely to the system of axioms for the characteristic function.

The proofs of soluability for sufficiently broad classes of games are rather complicated. Von Neumann and Morgenstern have ascertained (see 1.60.4.2) that any zero-sum four-person game has at least one solution. For the case of zero-sum five-person games, however, the question already remains open.

3. Shapley /1/ has offered another system of axioms free of the deficiencies mentioned. First of all, he has examined imputations, i.e., vectors fulfilling axiom 1° quoted above. Further, for every characteristic function v, among all imputations he has chosen such imputations

$$\phi(v) = (\phi_1(v), \ldots, \phi_n(v)),$$

which he has called <u>vectors of values of the game</u>, satisfying the following axioms:

2° For every automorphism *) ψ of the characteristic function v

$$\phi_{\psi i}(v) = \phi_i(v).$$

3° If player i is a "dummy" *), then

$$\phi_i(v) = v(i).$$

4° For any two characteristic functions v_1 and v_2

$$\phi(v_1 + v_2) = \phi(v_1) + \phi(v_2)$$

(It is not difficult to convince oneself that the sum $v_1 + v_2$ of two charact-

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*) For the definition of the automorphism (symmetry) of the characteristic function see page 256 f.

*) Player i is called a "dummy" if for any coalition S such that $i \notin S$, $v(S \cup i) = v(S) + v(i)$ holds.

eristic fun ions is itself a characteristic function.)

This system of axioms is consistent and complete. In particular, the following statement is true: For every characteristic function v there exists one and only one vector of values ϕ (v); the components of this vector are determined by the formula

$$\phi_i(v) = \sum_{S \subset I} \frac{(|S| - 1)!\ (n - |S|)!}{n!} (v(S) - v(S - (i)))$$

($|S|$ is the number of the elements in the set S).

This statement can be interpreted in the following natural way: There exists an interchange of players $\varphi : \{\varphi_1, \ldots, \varphi_n\}$. Somewhere in this interchange is player i. Put $i = \varphi_k$ and $S = \{\varphi_1, \ldots, \varphi_k\}$. Then the difference

$$v(S) - v(S - (i)) = \Delta(v, i, \varphi)$$

can be considered as an increase of the value of the characteristic function v at the cost of player i joining the coalition S - i under conditions of the interchange φ. Obviously, at different interchanges this increase may be different too. Shapley has proved the following theorem: The value ϕ_i (v) is the mathematical expectation of the increase $\Delta(v, i, \varphi)$ if all interchanges φ are equally probable.

In one of his later papers /5/ Shapley has extended this approach to a class of games with an infinite set of players and has obtained very interesting results.

4. Von Neumann and Morgenstern have systematically used the reduced form of the characteristic function for which $v(i) = -\gamma$ and $v(I) = 0$, usually assuming $\gamma = 1$. Such characteristic functions are now called -1 -0 - reduced characteristic functions. Sometimes one obtains an intuitively more appealing picture by passing to the 0 - 1 -reduced form of the characteristic function where $\gamma(i) = 0$ and $\gamma(I) = 1$. It is clear that it is possible to pass from one form to the other by means of the transformation of strategic equivalence.

§7: Zero-Sum Four-Person Games.

1. Contrary to three-person games forming only two classes of strategically equivalent games, four-person games are extremely manifold. Among them there is a continuous set of strategic equivalence classes described in a natural way by the points of a cube. Von Neumann and Morgenstern have found solutions for games corresponding to some domains of this cube. Complete sets of solutions have been shown only for the corners of the cube.

The types of solutions which thereby have been revealed are extremely manifold. It suffices to stress that for the main diagonal of the cube lying on the line $x_1 = x_2 = x_3$ there correspond solutions of various kinds to each of its parts $-1 \leq x_1 \leq -1/5$, $-1/5 < x_1 \leq 0$, $0 \leq x_1 < 1/9$, $1/9 \leq x_1 \leq 1/3$, $1/3 < x_1 < 1$.

The authors have postponed the detailed description of all results obtained to a subsequent publication (see the footnote on the page 305), but their plan has not been realized.

2. The immediate continuation of von Neumann's and Morgenstern's investigations in the field of the theory of four-person games has been undertaken by Mills /1/: He has enumerated all solutions of games corresponding to the faces of the cube and has shown some properties of games corresponding to the points of its edges.

An exhaustive analysis of a series of questions concerning zero-sum four-person games examined as quota games is due to Shapley /2/ (see III.3.6).

§8: Some Remarks Concerning $n \geq 5$ Participants.

1. The table of Figure 65 in I.19.2.3 (see page 331) is very impressive. It discourages one to engage in the cataloguing of solutions to five-person games (the classes of strategically equivalent five-person games form a ten-parametrical family!). Even classes of symmetric five-person games form a continuous family, though a one-parametrical one. Here it is appropriate to remember the regrettable circumstance that it is unknown whether all five-person games possess solutions.

It becomes particularly obvious that we must make the transition from the systematic description of games with few participants, to the development of an, in some sense more simply constructed, general theory of "calculation of games", where the study of properties of some games would lead to the study of properties of other games. The first example of such a reduction can be found in I.35.2, where the four-person game has been "split" into a three-person game and a separate "dummy". The second example is the analogy pointed out in 1.40.3, between the symmetric five-person games and the four-person games corresponding to the points of the main diagonal of the cube. One of the possible lines of development of these ideas has been discussed in chapter IX.

Another way consists in separating some classes of games possessing properties which facilitate their classification and analysis. (It is already clear that a fixed number of the players does not belong to these properties). One such class has been examined in chapter X.

2. In 1.40.2.3, the authors have made a very interesting remark about the description of games by means of formulating the aims of the players after their entering into a coalition. This reveals some perspectives of a reduction from the vague and descriptive cooperative aspect of the game to a more exact and formalized aspect - the strategic one. The important step in this direction has been made by the authors in §26.

§9: Composition and Decomposition of Games.

1. In chapter IX, von Neumann and Morgenstern have introduced the operation of composition on the set of all games (by the way, it would be particularly appropriate here to speak not about the set of games, but about the set of all characteristic functions). As in the whole book, the authors have avoided any evident refereneces to mathematical (in this case - algebraical) analogies, and only the mention of the book by G. Birkhoff and S. MacLane (on page 340) points to those algebraical associations which the authors possibly had in mind, and which they probably even followed.

The construction of the composition of games reminds us very much of direct products of groups, annulling sums of halfgroups, orthogonal products of spaces - in one word, those formations where the elements of various components interact with each other in the simplest manner. For games, the simplest interrelation among the players is the aimlessness of the coalitions formed by them (what can formally be expressed by the additivity of the corresponding values of the characteristic function).

2. In 1.44.3.2 the important question of the relationship between the mathematical propositions obtained by formal means on the one hand, and the requirements of "common sense" on the other hand, has been touched upon. The authors have been inclined (in the given place and generally) to attribute to "commone sense" the role of a criterion of truth, losing sight of its inevitable limitations and occasional subjectivity.

The requirements of common sense indeed play an essential role in establishing the fundamental theses of a mathematical theory. If,however, paradoxical conclusions of the theory are in contradiction to common sense, then one should not assume a priori that this circumstance overthrows the theory: Perhaps it testifies only to the necessity of revising or at least specifying the conception of common sense. The contemporary scientific common sense is differently used in case of non-commutative operations, non-Euclidean geometry, relativistic mechanics etc.

For instance,in certain cases,the incorrectness of statement (44:D) is not at all less in accord with common sense than, say, the possibility for automorphisms of the direct product of groups not to be the products of automorphisms of the groups. Besides, in the given case it turns out that the "intuitively probable" statement (44:B:a) is false too.

What has been said cannot diminish the pedagogical role of common sense, especially in such young disciplines which, like game theory, possess broad fields of application. Tha appearance of results, in early stages of the logical development of the theory, which are inconsistent with generally accepted intuitive conceptions can only uselessly discredit the theory. One can hope that in the course of time the propositions of game theory (concerning, for instance, the solutions in the cooperative theory) will be subjected to an objective experimental examination. The first steps in this direction have already been done (see for instance the papers by Kalisch, Milnor, Nash and Nering /1/, Luce /1/, Maschler /2/, Fouraker's survey containing an extensive bibliography, and among recent papers the report by Furst /1/).

For the time being, however, it is somewhat early to speak of systematical experimentation in game theory. Therefore it is completely understandable that here, as well as in numerous other passages, von Neumann and Morgenstern have not missed a chance for emphasizing the accordance of the conclusions of game theory with common sense.

At the same time, if one has no doubts about the truth of the theory, the divergence between theoretical conclusions and conceptions of common sense sometimes motivates modificaticns towards a generalization of the theory in order to enclose in its conclusions those intuitive considerations which originally have not been covered. So, the lack of naturalness of the conclusions about the solutions of the composition of games comes from the limitation to zero-sum games. The removal of this restriction (in 1.44.4) allows us to construct, in what follows, a more natural (though an essentially more cumbersome) variant of the theory.

3. The introduction of the excess (1.44.5 ff) for extended imputations involves the concept of an imputation as a possibility alternative to actually participating and acting in the game. Therefore, a too great negative excess stimulates the participation of the players in the game even without cooperation. A too great positive excess offers the players imputations more preferable than their possible payoffs are (even under conditions of coalition formation). Thus the investigation of the question of decomposition of zero-sum games leads to the necessity of the study of zero-sum games.

We note, at last, that the examination of detached (correspondingly, fully detached) imputations anticipates the subsequent introduction of the concept of the core for the characteristic function (see III.3.8).

§10: Simple Games.

1. The characteristic functions of simple games are intuitively defined in 0-1-reduced form (see II.6.4) as characteristic functions assuming only the values 0 and 1. Here the coalition S with $v(S) = 1$ is winning and with $v(S) = 0$ losing. In simple games the cooperative aspect of the game reaches its most accurate expression: In them, as von Neumann and Morgenstern have noted ingeniously, there is only one type of payoff.

An important class of simple games are the weighted majority games where to every player i, a weight w_i is attributed, and those coalitions are declared winning for which $\sum_{i \in S} w_i > 1/2 \sum_{i \in I} w_i$. Every such game is denoted by the symbol $[w_1, \ldots, w_n]$. It turns out that all simple games with a number of players not more than five are weighted majority games (for a large number of players that does not hold any more).

Heuristic reasonings led the authors to detaching some specific solutions of simple games calling them "main". The idea of the main solution consists in considering such imputations as acceptable where the members of some minimal winning coalitions share a unit with each other (supposing that the game is given in 0-1-reduced form). The actual obtaining of the main solution, however, turns out to be very complicated both logically and technically. The further development of these ideas is contained in a paper by Isbell /1/.

The theory of simple games has turned out to be very substantial. For results obtained in this direction see III.3.7.

§11: General Non-Zero-Sum Games.

1. The abandomnent of the constancy of the sum of the player's payoffs (and *a fortiori* of its transformation to zero) makes it necessary for von Neumann and Morgenstern to extend the theory constructed earlier. In principle, this can be done in several ways.

The first one of them is connected with a radical abandonment of the reduction of a game to its characteristic function and with the return to the primary concept of game theory - the payoff function. Here, however, the necessity arises of establishing more general principles of rational behavior of the players than those on which the theory of antagonistic games has been based. The rudiments of a corresponding theory emerged in the paper by Nash /2/, but a sufficiently complete conception of a "solution" of games in this sense does not exist up to now.

The second way is based on the formal extension of the results of the theory of characteristic functions to functions possessing only properties of standardization and supperadditivity

$$v(\emptyset) = 0 \qquad (1)$$

$$v(S \cup T) = v(S) + v(T) \qquad (2)$$

for any disjoint coalitions S and T.

Obviously, here the concept of an imputation as a vector $\alpha = (\alpha_1, \ldots, \alpha_n)$ can be defined such that

$$\alpha_i = v(i), \qquad \sum_{i \in I} \alpha_i = v(I),$$

and on the basis of it the concepts of domination of imputations and of solution.

Thus, the bond to the initial strategic aspect of the problem has been essentially weakened: The value of the characteristic function has been changed from the value of a real antagonistic game, to the value of some fictitious game, and even to a purely normative characteristic of the coalition. We note here that, as Gilles /2/ has shown, it is formally possible to drop the condition of superadditivity (2): For any arbitrary (i.e., not necessarily superadditive) characteristic function, a superadditive characteristic function can be found such that there exists a one-to-one domination-preserving correspondence between the imputations of the former and the latter.

2. Von Neumann and Morgenstern have chosen the second way, but for the purpose of preserving the intuitive meaning of the characteristic function they have undertaken a roundabout manoeuvre. For every general game (i.e., a game with a variable sum of payoffs Γ), they introduce a zero-sum game $\overline{\Gamma}$ which may be called an extension of the original game Γ to a zero-sum game, and which has to reproduce all its fundamental strategic and cooperative features.

The transition from the game Γ to its zero-sum extension $\overline{\Gamma}$, can be achieved by the inclusion of an additional fictitious player in Γ possessing a single strategy and in all situations automatically absorbing the entire algebraic sum of the losses of all remaining players.

The characteristic function of the game $\overline{\Gamma}$ can be determined on the basis of the theory developed earlier. After restricting the characteristic function obtained to the set of only real players, we arrive at the characteristic function of the game Γ possessing the properties (1) and (2).

From the purely strategic point of view, there is no difference between the initial game Γ and its extension $\overline{\Gamma}$ obtained: In both games the strategic possibilities of the players are the same and their identical use leads to identical results.

On the cooperative level, however, these games differ from each other essentially. Inasmuch as the fictitious player may obtain a larger or smaller payoff, he has some interests. For the realization of his goals, he can come to an agreement with other real players, compensating them for the losses they may suffer when increasing his payoff. This appears more probable as the strategic possibilities of the real players lose their meaning as a result of the transition to the characteristic function, and in that sense the real players themselves turn into fictitious ones. Such an equalization of the rights of the fictitious player with the real ones is shown intuitively in the example of 1.56.4.1.

Since the game $\bar{\Gamma}$ has been introduced for modelling the game Γ, it must be examined only partially, allowing in it only those possibilities of players which Γ possesses. This can be achieved by an appropriate modification of the concept of domination (resulting in the fact that the effectivity of every set is meant as the effectivity of the subset of all its real players), and the modified concept of a solution based on it (1.56.12). In this new solution, only such imputations are contained where the fictitious player, $n + 1$, obtains only $v(n + 1)$. Thus in the solutions of the initial game Γ, the cooperative possibilities of the fictitious player are disregarded.

3. The dependence of the payoff of each player on strategies selected by other players makes up the substance of game theory. Therefore, those cases are extremely interesting where this dependence is not valid to a full extent. In 1.57.4, von Neumann and Morgenstern have detached the extreme case of the absence of such dependence by introducting the concept of a removable set. The authors have pointed out that any two-element set in the essential zero-sum three-person game can be removed. The fact seems to be paradoxical, and therefore we will use the following example as an illustration.

Let player 1 possess two strategies and the sets of strategies of players 2 and 3 be arbitrary. If 1 plays his first strategy, then, independently of the strategies chosen by the players 2 and 3, he, as well as 3, receive a zero-payoff, and player two a unit. If 1 choses his second strategy, then 3 receives a unit in all situations, and 1 as well as 2 obtain zero.

This is an essential zero-sum three-person game in 0-1-reduced form, where the set of players $\{2,3\}$ is removable.

4. Von Neumann and Morgenstern devote the last four paragraphs of chapter XI to the economic interpretation of some simple facts of cooperative game theory. Here it should be emphasized once more that the contents of these sections deal neither with the description of any economic phenomena as such, nor with their modelling in terms of game theory, but solely with an interpretation using economic language of certain game-theoretical propositions. The authors are speaking about various kinds of markets: with one seller and one buyer (§ 61), with one seller and two buyers (§ 62 and § 63), and eventually a market with l sellers and m buyers (§ 64). As matter of fact, they have limited themselves to the examination of purchasing and selling acts only, considering in detail that case where only one transaction is possible. It is clear that such a game-theoretical analysis does not describe the phenomena mentioned in their full scope, but only their single features.

§12: Extension of the Concepts of Domination and Solution.

1. The differences in the variants of the concepts of domination and solution have motivated the authors to examine these questions in an abstract way, a way that does not fix the nature of imputations and assumes domination to be an arbitrary relation on the set of imputations. In particular, they have proved (in 1.65.8.2) the proposition which is a special case of Richardson's theorem (see I.2.6). The questions of the existence of somewhat modified solutions have been examined in Richardson's paper /3/.

2. In 1.66.3, von Neumann and Morgenstern have attempted to avoid the assumption of the transferability of utility. They have introduced domains of individual utilities $\mathcal{U}_i$ for every player i, and $\mathcal{U}(T)$ for every coalition T, forestalling thereby those results that have been obtained only recently. (see III.3.11.).

3. An important assumption of the entire theory is the condition of unrestricted divisibility of utility. The abandonment of this condition leads to various consequences: In 1.65.9.2, the possibility of an approximation of games with continuous sets of imputations and a multitude of solutions for every game by games with discrete sets of imputations and a unique solution for every game, has been pointed out.

The limited divisibility of utility can be interpreted as a limitation of a person's ability of distinction of utilities, i.e., as a limitation (in the well-known sense) of the information a person possesses about his utilities. The assessment of such an imperfection of information is the result of a consistent game-theoretical approach to this question. As was to be expected it, turns out (see 1.67.3) that the player who is able to make finer distinctions of utilities is in an advantageous position.

C h a p t e r III

T H E O R Y O F G A M E S - A B R A N C H O F M A T H E M A T I C S

Since the time the monograph Theory of Games and Economic Behavior was published, game theory has developed into an extensive discipline. A considerable number of monographs have been dedicated to it, which are discussed in sufficient detail in O. Morgenstern's foreword to the present edition, and an enormous number of articles (the bibliography compiled in 1953 by G. Thompson and D. Thompson /1/ has already comprised 1009 titles).

The variety of several lines of development is characteristic of contemporary game theory. That is completely natural, for decision-making both in conflict situations and under conditions of uncertainty is an extremely complicated subject whose study can be approached from various sides. Therefore, a description, or at least a systematization of all results obtained in game theory, is practically impossible. In this chapter the fundamental trends in game theory will be analyzed and the most typical results will be presented.

As we speak here about game theory as a branch of mathematics, we will not touch the various applications of game theory, despite their interest and instructive content.

§1: Matrix Games.

1. At the present time, matrix games are sufficiently studied from a theoretical point of view. Bohnenblust, Karlin, and Shapley /1/ have ascertained the possible location of the polyhedra of optimal strategies of the players in matrix games within the simplices of all their mixed strategies, showing the necessary relations between the dimensions of the former and the latter. They have given a method for constructing a matrix game with a given solution. The investigation of this last problem has been completed in the paper by Gale and Sherman /1/, who have characterized the set of pairs of convex polyhedra which can serve as the sets of optimal strategies for the players in some matrix game, and have also found a method for the description of all games with given sets of optimal strategies for the players. For matrix games, the plurality of solutions to games is in some sense an exception. Bohnenblust, Karlin, and Shapley /1/ have ascertained that the set of all mxn-games with unique solutions is open and dense in the mn-dimensional Euclidean space of all mxn-games.

Extremely interesting investigations concerning the general combinatorial properties of saddle points in matrices have been carried out by Shapley /6/.

2. The considerations in 1.18.2 of von Neumann's and Morgenstern's book show that the description of solutions of matrix games by one single formula is practically impossible even in the simplest cases. Hence, there are two natural ways for finding solutions of games.

First, one can attempt to detach some classes of matrix games which depend on a small number of parameters (for this see II.4.2). Solutions for such (by the way, pretty narrow and not numerous) classes of games can be found in books by Karlin /3/ and Dresher /1/.

Secondly, one can construct an algorithm for finding all (or at least some) solutions to an arbitrary matrix game.

The first such algorithm possessing also generally theoretical significance has been found by Shapley and Snow /1/. Its essence can be described by the following theorem.

Let X and Y be optimal strategies of the players I and II in a game with payoff matrix $A \neq 0$. A necessary and sufficient condition for X and Y being extreme points (corners) of the corresponding polyhedra of optimal strategies, is the existence of a non-singular $r\times r$-sub-matrix B of A for which

$$X_B = \frac{J_r B^{-1}}{J_r B^{-1} J_r^T}, \quad Y_B^T = \frac{B^{-1} J_r^T}{J_r B^{-1} J_r^T}, \quad v(A) = \frac{1}{J_r B^{-1} J_r^T}. \qquad (1)$$

(Here X_B and Y_B denote those parts of vectors X and Y which are composed of the elements corresponding to the rows and columns of A that are contained in its sub-matrix B; J_r is a r-dimensional vector all components of which are equal to unity).

As every finite matrix contains only a finite number of sub-matrices, the successive examination of all sub-matrices of A, the formation of vectors X_B and Y_B in accordance with formula (1), and the examination of their "strategic qualities" and "optimality", allows us to find all the corners of the polyhedra of optimal strategies after executing a finite number of operations. Hence, passing to convex combinations may result, in general, in the discovery of all optimal strategies too.

3. A very interesting method for the approximative finding of optimal strategies of players in a matrix game based on differential equations has been offered by Brown and von Neumann.

Let A be a skew-symmetric nxn-matrix (an examination of games only with skew-symmetric payoff matrices does not diminish the generality of the considerations), then an optimal strategy for each of the players is a limit point of the solution of the system of differential equations

$$\frac{dx_i(t)}{dt} = \varphi_i(x) - x_i \sum_{j-1}^{n} \varphi_j(x) \qquad (2)$$

for arbitrary starting conditions $(x_1^\circ, \ldots, x_n^\circ)$, such that

$$x_i^\circ \geq 0, \quad i = 1, \ldots, n; \quad \sum_{i-1}^{n} x_i^\circ = 1,$$

where

$$\varphi_i(x) = \max \{ 0, \sum_j a_{ij} x_j \} .$$

Unfortunately, in practice even the numerical solution of system (2) causes great difficulties.

4. Another way of finding at least one strategy for each player, offered by Brown and strictly proved by J. Robinson /1/, consists in applying an iterative procedure. This can also be interpreted as an experimental finding of optimal strategies in the course of a match each of whose plays consists in playing the given matrix game, where in each play, each player fixes the

frequencies of his adversayr's choice of pure strategies for the time elapsed since the match began regarding them as the probabilities in a mixed strategy, and chooses that pure strategy which is the best against this mixed one.

The estimation of the convergence of the described process, given by Shapiro /1/, shows that it converges very slowly: The error in the determination of the value of an nxm-game after t steps is of the order $t^{-\frac{1}{n+m-z}}$.

5. As Gale /1/ has written, "one of the most striking events in connection with the emergence of modern linear economic model theory was the simultaneous but independent development of linear programming on the one hand, and game theory on the other, and the eventual realization of the very close relationship that exists between the two subjects". According to Dantzig's evidence /1/ the relationship between linear programming and game theory has been already pointed out by J. von Neumann in 1947. Later on Gale, Kuhn, and Tucker /1/ have occupied themselves with these questions. In the most natural form, the equivalence of a pair of dual problems of linear programming and a matrix game is given by the theorem of Dantzig and Brown, and published by Dantzig /2/. Consider a pair of dual problems of linear programming

$$\begin{array}{ll} AX^T \leq B^T, & YA \leq C, \\ CX^T \to \max, & YB^T \to \min, \\ (X \geq 0), & (Y \geq 0). \end{array}$$

A necessary and sufficient condition that $X°$ and $Y°$ are optimal solutions of these problems is that the vectors

$$(tX°, tY°, t)$$

is an optimal strategy for one of the players in the game with the matrix

$$\begin{pmatrix} 0 & -A & B^T \\ A^T & 0 & -C^T \\ -B & C & 0 \end{pmatrix}$$

(since this matrix is skew-symmetric it is not important which player in particular is here referred to), where $t > 0$.

The equivalence of the solution of a matrix game with the solution of problems of linear programming allows us to utilize all methods of solution developed for the latter problem for solving the former. In particular, optimal strategies in matrix games can be found by means of the well-known simplex-method. A very practical method for solving a matrix game by means of linear programming has been proposed by Tucker /1/.

6. The correspondence discovered between matrix games and pairs of dual problems of linear programming is that to every class of matrix games belongs a class of problems of linear programming, and vice versa. It may turn out that, to a class of games naturally delineated in the sense of the intuitive interpretation of the games forming it, there corresponds a class of pairs of linear programming problems also delineated in a sufficiently natural way, but already in another, "linear programming" interpretation. Examples for such a correspondence between the classes may prove very useful because certain facts considerably tangled in one of these classes may be intuitively clear in the other one.

One such example has been examined by von Neumann in paper /2/. It consists in the following.

Given an mxn-matrix $A = ||a_{ij}||$. Player I has cells (ij) as his strategies and player II its rows (i) and columns (j). If player I chooses the cell (ij), then his payoff for player II's choice of row i or column j is equal to $-a_{ij}$, otherwise it is equal to zero. In the result we obtain a mnx (m + n)-matrix game.

All games of this type can be interpreted as follows. Player I "hides" in a cell of matrix A and player II attempts to guess the row or column on which this cell is located. If player II succeeds in guessing it, then he receives from player I an amount equal to the number in the cell of the matrix chosen.

This game is equivalent to the linear programming problem consisting in maximization of the sum

$$\sum_{i,j} a_{ij}x_{ij}$$

for non-negative x_{ij}, subject to the constraints

$$\sum_{i=1}^{m} x_{ij} \leq 1, \ j = 1, \ldots, n.$$

$$\sum_{j=1}^{n} x_{ij} \leq 1, \ i = 1, \ldots, m.$$

Such a problem of linear programming is known under the name "assignment problem": The problem of distributing, in an optimal way, m persons to n jobs if the effect of person i in job j is equal to a_{ij}.

7. If the set of strategies of the players in an antagonistic game are convex polyhedra (with a finite number of corners) in finitely dimensional Euclidean

spaces, and the payoff function is bilinear, then the game is called <u>polyhedral</u>. Since strategies are like "physical" mixtures of corners, the interior points of the strategy polyhedra result, in view of the bilinearity of the payoff function, in the same payoffs as the corresponding probability mixtures, i.e., mixed strategies. Therefore, in polyhedral games, one can confine oneself to the examination of only those pure strategies which are located in the corners of the strategy polyhedra, thereby changing the polyhedral game into a finite one.

The game obtained turns out to be a matrix game if the original polyhedra of strategies are given by their corners. If, however, these polyhedra are given by their edges, then the game remains now, as before, a finite antagonistic (i.e., matrix) game, but in order to derive its payoff matrix in an obvious form, it is necessary to put up with well-known inconveniences which can often be rather considerable. Therefore, the question arises about the solution of polyhedral games without this preliminary reduction to the traditional matrix form. Obviously we meet here a sub-case of the problem touched upon in II.4.2.

The relationship between matrix games and linear programming allows us to solve this problem, as has been done in paper by Dantzig /3/, Charnes /1/, and Wolfe /1/.

Let the bilinear form which determines the payoff function of a game have an mxn-matrix A (thereby we assume that the polyhedra of strategies of the players are located in m- and n-dimensional vector spaces respectively) and the polyhedra of strategies by given by their edges, i.e., by systems of inequalities of the form $BX^T \leq b^T$, $YC \geq c$ (here B is a kxm-matrix, C an nxl-matrix, and a and b are k- and l-vectors respectively; k and l obviously denote the numbers of edges of the strategy polyhedra). Then the optimal strategy of player I can be obtained as a part X of the optimal solution (X,V) of the linear programming problem,

$$\begin{aligned} BX^T &\leq b^T, \\ -AX^T + CV^T &= 0, \\ V &\geq 0, \end{aligned}$$

and, similarly, the optimal strategy of player II as a part Y of the optimal soltuion (Y,U) of the linear programming problem

$$\begin{aligned} YC &\geq c, \\ -YA + UB &= 0, \\ U &\geq 0. \end{aligned}$$

8. Polyhedral games appear sometimes in the form of games with linear constraints, i.e., of such matrix games where the players can not apply all their mixed strategies, but only those whose probabilities are subject to some linear inequalities. In this case, the space of pure strategies of the players in the polyhedral game obtained will be a convex polyhedron located in the simplex of all its mixed strategies of the original matrix game.

As a simple example which is, however, of principal importance, we show a matrix game where player 1, according to some considerations known to his opponent, plays in any of his mixed strategies the first pure strategy at least as often as the second, the second at least as often as the third etc. That means, that any mixed strategy $x = (x_1, \ldots, x_n)$ feasible for him, has to satisfy the conditions

$$x_1 \geq x_2 \geq x_3 \geq \cdots \geq x_n.$$

In the given case, the inequalities describe a smaller simplex within the simplex of all mixed strategies of player 1 and therefore the payoff matrix of the new game can be described without difficulties.

§2: Infinite Antagonistic Games.

1. Infinite antagonistic games (as well as all antagonistic games) are given by defining spaces A and B of strategies of the two players and a payoff function H on the product A x B. Therefore, it is natural to write any such games as a triplet $\langle A, B, H \rangle$. The maximin principle of behavior of players in antagonistic games completely characterizes their behavior from the point of view of the maximization of the mathematical expectation of the payoff. At the same time, if a game has many solutions (equivalent in the sense shown), then the question arises as to which of them is the best with respect to some other, additional point of view. As such a by-principle of optimal behavior, Buck /1/ has proposed the best exploitation of mistakes by the adversary. The same criterion has been examined by Huyberechts /1/; she has analysed a principle based on the minimization of a characteristic of deviation of the payoff function from the value of the game.

The maximin principle of behavior of players does not depend on the cardinality of the sets of the players' strategies. Therefore the transition from matrix games to the infinite antagonistic games does not cause any conceptual, purely game-theoretical, difficulties. The question of the existence of the value of a game and its solutions, however, look different when passing to the infinite case.

2. As we know, it is advisable in games of strategy to employ mixed strategies, i.e., probability distributions on the sets of pure strategies of the players. However, in order for the examination of mixed strategies in a game $\langle A, B, H\rangle$ to make sense, the existence of the payoff, under conditions generated by any pair of such strategies of the players, is necessary, i.e., the integral

$$\iint\limits_{AB} H(a,b)\,dF(a)\,dG(b)$$

must exist for any pair of mixed strategies F and G.

This question is substantial because of the following reason:

The examination of the probability measure F, on the set A itself, signifies a preliminary introduction on A of a σ-algebra $\mathfrak{A}$ of subsets measurable by F. Likewise, the examination of the measure G is connected with the σ-algebra $\mathfrak{B}$ of subsets of B measurable by G. The examination of the integral (3) is connected with the σ-algebra $\mathfrak{L}$ of subsets generated by all cartesian products of the form of $K \times L$, where $K \in \mathfrak{A}$ and $L \in \mathfrak{B}$. If the payoff function H is measurable on the σ-algebra $\mathfrak{L}$, then the integral (3) does exist.

3. The prototype of numerous existence theorems for infinite antagonistic games is Ville's theorem /1/ on the strict determinateness of games on the unit square with a continuous payoff function. In fact, in proving this theorem, the only fact used is that the segments of strategies of the players in their usual Euclidean topology are conditionally compact spaces*) and the payoff

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*) A space is called <u>conditionally compact</u> (in another terminology,totally bounded), if from any infinite sequence of its elements a sub-sequence can be chosen that converges to an ele-ent of this space.

function is continuous in this topology.

Thus, it turns out to be quite essential what topology (or in particular, what metric) has been assumed on the strategy sets of the players.

4. Obviously, the distance defined between strategies has to reflect the distinction between them. This distinction can be purely "physical", describing the difference between them as activities actually realized by the players, but a player's choice of strategy is made not on the basis of its external, "physical" characteristic, but for the sake of the maximization of his payoff.

Wald /2/ has introduced an <u>intrinsic</u> metric (which may be called also uniform or Chebyshev's metric; Wald himself has sometimes called it also Helly's metric) on the sets of strategies, defining in the game $\langle A, B, H \rangle$ for any $a', a'' \in A$

$$\rho_A(a', a'') = \sup_{b \in B} \left| H(a',b) - H(a'',b) \right|$$

and analogously for the space of strategies of the second player. Obviously, in the usual (intrinsic) topology, the payoff function is continuous in each of its variables (i.e., in the strategies of each player).

Let $\mathfrak{A}$ and $\mathfrak{B}$ be σ-algebras of subsets of A and B generated in the usual topology by open subsets of the spaces A and B. Denote by $\mathfrak{L}$, the smallest σ-algebra which contains any subsets A x B of the kind K x L, where $K \in \mathfrak{A}$ and $L \in \mathfrak{B}$. As Wald /2/ has shown, the payoff function H is always measurable on $\mathfrak{L}$ if at least one of the spaces A and B is separable in its usual topology.

Wald has proved /2/ the strict determinateness of infinite antagonistic games where the strategy spaces of the players are conditionally compact in their usual topology (it is remarkable that from the conditional compactness of the strategy space of one player, follows the conditional compactness of the strategy space of the second player). He has ascertained that if, instead of the conditional compactness of strategy spaces, their compactness is required, then this theorem can be strengthened: Instead of the strict determinateness, the existence of optimal strategies of players can be confirmed. For all its seeming intuitiveness, this result is extremely subtle and non-trivial.

5. Unfortunately, in several cases the usual topology turns out to be too "rough". For instance, if in a game on the unit square, the payoff function is discontinuous in every point of the diagonal of the square (in remaining points it may be also continuous), then the distance between any two points is larger than a fixed positive number such that the strategy spaces of the players in the usual topology consists everywhere of isolated points. Therefore, along with the usual topology, other methods of topologizing the strategy spaces of the players are also utilized. Karlin /1/ has introduced a weak topology where the sequence of strategies f_1, f_2, ... of one of the players (let it be the first) is convergent to the strategy f, if for any strategy g of the second player

$$\lim_{n \to \infty} H(f_n, g) = H(f, g).$$

It is clear that under conditions of a weak topology, the necessary properties of the space of strategies are achieved for broader conditions concerning the payoff function, than under conditions of the usual topology.

Therefore, the examination of the weak topology results in stronger assertations. In particular, Karlin has proved the strict determinateness of the game on the unit square, if the points of discontinuity of the payoff function coincide with the diagonal.

A very general theorem on the existence of values of antagonistic games has been proved by Wu Wen-tzun /1/.

6. Yet since the time of Ville's paper /1/, examples of antagonistic games have been known which have no values. So the game has no value where the sets of strategies of the players are sets of natural numbers and the payoff function is defined by the relation

$$H(m,n) = \begin{cases} 1 \text{ for } m > n \\ 0 \text{ for } m = n \\ -1 \text{ for } m < n \end{cases}$$

A natural approach to games of this type consists in a further extension of the concept of strategy which includes the concept of mixed strategy as a probability measure on the sets of pure strategies. This approach leads to employing finitely additive measures on the sets of pure strategies as new, generalized strategies, of the players. The set of finitely additive measures turns out to be sufficiently universal: In finitely additive measures as strategies every game with a measurable bounded payoff function has a value. The idea of this result comes from the paper by Karlin /1/, and the strict proof belongs to E.B.Yanovskaya /2/. In her paper /1/ she began for the first time to examine games with unbounded payoff functions.

7. Except for existence theorems, there are perhaps, no other propositions in game theory that are true for all games from sufficiently broad classes. Moreover, even in the several examples of infinite games for which some non-trivial results have been obtained, the sets of strategies are either subsets of finitely dimensional Euclidean spaces (or of probability measures on such spaces) or sets of finitely dimensional vectors each component of which is a bounded function defined on a given subset of Euclidean space. Games on the unit square obviously belong to the first of these classes. To them also the largest number of papers is dedicated.

8. Obviously, every matrix game can be considered as a polyhedral game and thereby as an infinite one. Conversely, the reason why the solution of polyhedral games is relatively so simple, is that every such game is determined by a finite number of parameters.

The next games, according to their complexity, are those on the unit square where the payoff function $H(x,y)$ is one of the form

$$\sum_{i=1}^{n} r_i(x) s_i(y) \tag{4}$$

Such games are called <u>separable</u>. From the point of view of the mathematical expectation of the payoff, every mixed strategy F of player I in a separable game with payoff function (4) can be described by its moments

$$\int_0^1 r_i(x)\, dF(x)$$

(the same holds also for player II. Therefore, mixed strategies in a separable game with payoff function (4) can be described with equivalent accuracy by n parameters. Properties of separable games have been examined in papers by Dresher, Karlin, and Shapley /1/,Dresher and Karlin /1/, and also Gale and Gross /1/.

An important stimulus for the development of the theory of separable, and particularly, polynomial games, has been the hope that with the aid of such games one could succeed in approximating arbitrary continuous games on the unit square. This hope did not come true however, for first the polynomial games themselves turned out to be extremely complicated and, secondly, the convergence of the polynomials to the continuous functions is, generally speaking, pretty slow. A systematic exposition of results obtained for this class of games is contained in Chapter 11. of Karlin's monograph /3/.

9. A great number of investigations have been dedicated to games of timing. A natural interpretation of these games as "duels", consists in the following. Let us suppose that each of two opponents can fire at his adversary in instant $t \in [0,1]$, where the accuracy of each player increases with time. A sensible consideration is that the player should fire sufficiently late (in order to fire with the greatest probability of a hit), but not too late (because his adversary can go before with his shot and kill him).

Several modifications of games of timing have been examined in numerous articles. Shiffman /1/, for the symmetric case, and Karlin /1/, for the general case, have reduced the finding of optimal strategies for players in games of timing to the solution of associated integral equations, and have found a way of solving such games. A detailed exposition of most of the results related to this class of games is contained in Chapter 13 and 14 of Karlin's monograph /3/

(especially see the notes and references to these chapters).

Games where one or both players have to choose several instants of time belong also to games of timing. Such a game can be interpreted as a duel with many shots. These games have been treated by Blackwell and Girschick /1/, Restrepo /1/, and others.

10. Games where all situations split into two classes, also belong to a class of simpler games on the unit square: Advantageous ones for player I, where he receives a payoff equal to one, and disadvantageous one for him, where his payoff is equal to zero. It can be assumed that the payoff functions in such games are characteristic functions of some sets.

One example of such a game with a military tactical interpretation has been given in Dresher's book /1/. For further examples of technical content see N.N.Vorob'ev's paper /7/.

11. Extremely interesting are those games where the strategies of the players are functions. Here, one has to take into account that the functional nature of strategies does not yet contradict the normal form of the game. For a game in normal form the instantaneous choice of strategies as a whole is characteristic, even if an entire function is chosen.

In a broad class of games of this type, the player selecting a function is called machine-gunner (in foreign literature also bomber). The function selected can be interpreted as the intensity with which he fires at every instant of time. Obviously, the strategy of the machine-gunner can be regarded as an extreme case of the strategy of a player choosing only a few instants of time. If his opponent chooses the time from an interval, then he is called sharpshooter (correspondingly - fighter). The analysis of games of this class requires the use of the Neyman-Pearson /1/ lemma, well-known from statistics. For a detailed examination of these games and the history of the problem, see chapter 16 of Karlin's monograph /3/.

12. A somewhat distinct place is occupied by games of the Poker type, treated in detail in § 19 of the monograph by von Neumann and Morgenstern. In these games, the strategy of the player is in fact not a concrete decision, but a function whose values are such decisions and whose domain is the set of states of information. Therefore, it would be natural to refer to Poker as a dynamic game. Fundamental results concerning the theory of games of this type are given in chapter 17 of the monograph by Karlin /3/.

§ 3: Cooperative Theory.

1. From the point of view of principles, antagonistic games have been examined exhaustively. The transition to a broader class of non-coalition games has been outlined, but not developed further, by von Neumann and Morgenstern, who have given a general definition of such a game, yet have not formulated general principles of rational behavior for players in such games. They have reduced the problem to the study of the characteristic function of a game, i.e. in the last resort to a system of antagonistic games.

This reduction, for all its conceptual profundity and analytical elegance, turns out to be substantially imperfect because it does not reflect all features of the original game. Very convincing evidence of the imperfection of this theory is seen in an example given by McKinsey /1/:Player I, choosing one of his strategies, receives the payoff 0, and his partner, player II, the payoff 10;

if I chooses his second strategy he obtains -1 000 and II obtains 0. Here, obviously, $v(I) = 0$; $v(II) = 0$; $v(I,II) = 10$. If we limit ourselves to the examination of only the characteristic function, then we inevitable omit some substantial features of the situation.

Besides, the classical cooperative theory of von Neumann and Morgnestern assumes the presence of individual, unrestrictedly transferable, and quantitatively invariant utility (the sole passages where the authors have attempted to abandon this assumption are § § 66, 67, in particular see 1.66.3). What has been said has predetermined the possibilities (to a considerable extent now already realized) of the whole hierarchy of generalizations of von Neumann's and Morgenstern's original cooperative theory.

We note above all, that it is not possible to retain the unrestricted transferability of utility in games, but abandon its invariance. As Aumann /1/ has shown, if at least three players are participating in a game, then from the unrestricted transferability of utility follows its linearity. Thus, if one abstracts from modifications without principal significance, von Neumann's and Morgenstern's theory covers all cases of unrestrictedly transferable utility.

The first possibility of generalization is connected with the development of the classical cooperative theory without transferable utility (i.e., without common "money" which is of "equal utility" for all players) but with side payments which are carried out by means of non-monetary goods; here, in fact, a transfer of utility takes place, but there does not exist a common scale of account for utilities and, a fortiori, the total increase in utility need not equal zero. The following step, in the direction of a generalization of the classical cooperative theory, completely abandons transfers of utilities, but maintains the general method of description of games that remains of the characteristic function. The corresponding theory is called thoery of games without side payments. Further, it is possible to revert to the purely strategic aspect of a game and to take into account that the expected payoffs of the players (or coalitions) are not primary data, but are calculated on the basis of the values of the payoff functions in various situations. On this level of generalization, the theory of non-coalition games has emerged.

Eventually, if one assumes that the elementary participants in the game (i.e., the parties to which the payoffs in various situations are ascribed) are coalitions which may intersect each other, then we obtain the broadest class of games, the class of coalition games.

Non-coalition games are often called non-cooperative. At this terminological distinction we have to halt because this is not a question of what kind of associations of players is to be called coalitions and which cooperations, but a question of the theory itself. If we extend the class of cooperative games studied, then we abandon the cooperative aspect as a constitutive form of rational actions of players, thereby coming to the non-cooperative games. If, conversely, we start from the most general, the coalition games, and limit ourselves to the case of one-element (or what is in fact the same, pairwise disjoint) coalitions, then we come to the same class of games which may now, however, naturally be called non-coalition games.

We turn now towards the exposition of fundamental results obtained at each stage of these generalizations. For convenience, we divide the entire material into two parts. A great number of positive inquiries into these problems as well as a critical examination of several approaches to them is contained in the monograph by Luce and Raiffa /1/.

2. The cooperative theory of coalition *) games has been elaborated by von Neumann and Morgenstern most thoroughly and carefully. About two-thirds of the total volume of the monograph is dedicated to it. Besides, this theory has brought forth an abundance of propositions which allow for a natural interpretation in economic and sociological terms. Therefore, one would have expected that in the following years the cooperative theory would have made considerable progress.

These expectations have been fulfilled only partially. Although the papers dealing with questions of the cooperative theory are mathematically subtle and original, their number has been, for a long time, rather small and has increased only recently. This circumstance has been caused by a few reasons.

The first concerns the glaring lack of tradition in the entire set of game theoretical problems. This can be explained by the fact all "traditional" mathematical theories concern themselves with the several aspects and variants of the motion of physical bodies in physical space, whereas game theory deals with aspects of rational and expedient behavior. This also explains the lack of tradition of the mathematical apparatus used in game theory. But if one has, nevertheless, succeeded in connecting anatagonistic games with classical problems of linear algebra, functional analysis, and integral equations, or with the not so classical but sufficently intuitive convex polyhedra, then the investigations

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*) Translator's note: In the original Russian text the author speaks here of "non-coalition games", which is, however, obviously a misprint.

of cooperative thoery are based only on intricate elementary combinatorial arguments which are of a very special nature in every case.

The second is the vagueness of the fundamental concept of game-theory - of optimal behavior under conditions which go beyond the scope of the purely antagonistic conflict as well as the difficulties (lest to say impossibility) of an experimental examination of its axioms. With this is connected the incompleteness of the cooperative theory concerning the choice of some solution from many, and the choice within the solution of a certain imputation.

Thirdly, antagonistic games, from the very beginning, have turned out to be very closely connected with applications, both inner mathematical, theoretical (for instance, to mathematical statistics) and applied ones (military tactical problems). Therefore, persons who have occupied themselves with game theory have been stimulated by those branches displaying more determinateness without having to do immediately with the cooperative theory.

In view of what has been said, the cooperative theory presently shows a pretty variegated picture. We will pay attention only to some of the most important phenomena.

3. Until recently the question of the existence of solutions for arbitrary cooperative games has been open. Lucas's /1/ example has given a negative answer to this question. Therefore new problems have arisen: The classification of insoluable games in accordance with the causes of their insolubility as well as the search for a new ("generalized") concept of solution that would exist for every game. Until the latter has been carried out, the principles of rational behavior (and solutions in the sense of von Neumann -

Morgenstern are something like that, in spite of their vulnerability for criticism) will prove to be limited. Games that are not covered by the exactly formulated principles, slip from the domain of mathematics into the domain of psychology, remembering once more Borel's point of view quoted in I.4.2.

At the same time, the search for the possibly broadest classes of games which do possess solutions remains topical. For the time being, only single, comparatively special, results have been obtained in this direction.

So Gillies /2/ has shown that a "positive fraction" of all games have solutions and a "positive fraction" of all non-zero-sum games have unique solutions.

An example of a contrary character has been constructed by Kalisch and Nering /1/. They have examined games with an infinite (countable) set of players I. Imputations ($\alpha_1, \alpha_2, \dots$) of such a game, if it is given in -1-0-reduced form, have to fulfill the following relations

$$\alpha_i = -1 \text{ for every } i \in I$$

$$\sum_{i \in I} |\alpha_i| < \infty$$

$$\sum_{i \in I} \alpha_i = 0$$

The game is called <u>finitely decreasing</u>, if for any coalition $S \subset I$ and player $i \in I$

$$v(S) \leq v(S - i)$$

(it is clear, that a game with a finite set of players cannot be a finitely decreasing one). A proof has been given that for finitely decreasing games no solutions exist.

The example of a game which has no solution constructed by Lucas /1/ has a very artificial character. In this example

$$I = \{1,2,3,4,5,6,7,8,9,10\} ,$$

$$v(I) = 5; \; v(1,3,5,7,9) = 4;$$

$$v(1,2) = v(3,4) = v(5,6) = v(7,8) = v(9,10) = 1;$$

$$v(3,5,7,9) = v(1,5,7,9) = v(1,3,7,9) = 3;$$

$$v(3,5,7) = v(1,5,7) = v(1,3,7) = 2;$$

$$v(3,5,9) = v(1,5,9) = v(1,3,9) = 2;$$

$$v(1,4,7,9) = v(3,6,7,9) = v(5,2,7,9) = 2;$$

$$v(S) = 0 \text{ for any remaining } S \subset I.$$

(The given characteristic function is not superadditive; in respect of what has been said in III.11.1, however, from the point of view of solutions this circumstance is not essential).

4. The variety observed, in the structures of several solutions of different games, is, as it has turned out, completely regular. Shapley /3/ has found out that whatever form the closed set I in n-dimensional Euclidean space may have, there exists a (non-zero-sum) n+3-person game where one of the solutions splits into two closed parts such that one of these parts is "similar" to the set I.

Thus, the individual description of solutions of games beyond the class of non-zero-sum four-person games (discussed in chapter VII. of the monograph), appear to be aimless.

General propositions about solutions of arbitrary games are of a rather limited character. For instance, Gillies /2/ has pointed out that imputations belonging to some solution cannot be located "sufficiently near" to the corners of the simplex of imputations.

Shapley /3/ has raised the question of the existence of solutions which are infinite but only countable sets of imputations depending also on the number of players, and the existence of an upper bound for the number of imputations in a finite solution. Galmarino /1/ has ascertained that four-person games have no countable solutions and that for finite solutions there exists the upper bound required.

5. The impossibility of giving the general properties of solutions valid for all games, leads in a natural way, to the separation of different classes of games and to the study of their respective solutions. The first step in this direction has been made in the monograph by von Neumann and Morgenstern, and a large number of papers have emerged afterwards dedicated to specific games characterized by particular intuitive characteristics. The formal description of such a class of games consists in the following.

Let the set of players I be given in the form of a union $P \cup Q$ of non-intersecting groups P (sellers) and Q (buyers) where for any coalitions S (market)

$$v(S) = \min\{|S \cap P|, |S \cap Q|\}$$

(i.e., intuitively the payoff of the market is equal to the number of contracts).

Shapley has described a class of solutions for a market model including the unique symmetric solution (in every imputation of which all components corresponding to the buyers as well as all components corresponding to sellers are equal to each other) *).

6. An important class of games, the so-called "quota games", has been investigated by Shapley in his paper /2/. By <u>quota</u> in a game with characteristic function v, we mean a vector $(\omega_1, \ldots, \omega_n)$ having the two properties:

$$v(i \cup j) = \omega_i + \omega_j \quad \text{for any } i, j \in I,$$
$$\omega_1 + \ldots + \omega_n = v(I).$$

It turns out that in order for a game (characteristic function) to possess a quota, it is necessary and sufficient that for any four distinct players i, j, k, l

$$v(i \cup j) + v(k \cup l) = v(i \cup l) + v(j \cup k)$$

holds, and besides that

$$\sum_{\substack{i \neq l \\ i, j \in I}} v(i \cup j) = 2(n-1)\, v(I).$$

From this follows, among other things, that any zero-sum four-person game possesses a quota. Shapley has proved, that every quota game possesses a solution (see i.60.4) and has given one of these solutions in an obvious way.

*) Thus, Shapley /4/ has systematically examined one of the cases of market models outlined by von Neumann and Morgenstern in §64 of their monograph.

This solution is based on the domination of two-player coalitions.

This idea of the quota, as well as results obtained by Shapley, have been generalized by Kalisch /1/. In particular, he has introduced an m-quota as a vector $(\omega_1, \ldots, \omega_n)$, such that $v(S) = \sum_{i \in S} \omega_i$ for any m-element coalition S and has pointed out the necessary and sufficient conditions for the existence of a solution of a game based on this m-quota.

7. Much interest has been attracted by simple games. Zero-sum simple games have to be proper (i.e., the coalition and its complement cannot be winning at the same time) and strong (i.e., the coalition and its complement cannot be losing at the same time). Going beyond the scope of zero-sum games leads to the possibility of emergence of improper and (or) non-strong simple games.

In a non-strong game, the losing coalition together with its complement is said to be "blocking". Elaborating ideas outlined by von Neumann and Morgenstern in §53 of their monograph, Richardson /4/ has proposed to discuss games as projective spaces, where points are regarded as players and pairwise intersecting subspaces of the least dimension as winning coalitions. He has obtained a series of results concerning the existence of blocking coalitions where the number of players has a certain arithemtical structure.

If a simple game has a transitive group of automorphisms, then it is said to be homogeneous. Not every number of players can participate in a strong homogeneous game. Isbell /2/ has shown that for every odd m, one can find an h such that for $k > h$ strong homogeneous games with $2^k m$ players do not exist. Extraordinarily important, interesting, and promising is the study of any kind of operations on sets of games allowing the construction of complicated and manifold games from single, comparatively simple, and uniform components. Compositions of games introduced by von Neumann and Morgenstern (in chapter IX) belong to operations of this type. For the time being, one has not succeeded in discovering other sufficiently natural constructions applicable to arbitrary games. Shapley /6/ has, however, introduced the concepts of sums and products of simple games.

Let $\Gamma(P_1, W_1)$ and $\Gamma(P_2, W_2)$ be two simple games in 0-1-reduced form with non-intersecting sets of players (here and further P_1, P_2 and P are the sets of players in the games, W_1, W_2 and W are the sets of winning coalitions respectively; the characteristic functions of these games are denoted by v_1, v_2 and v). A game is called the sum of two games

$$\Gamma(P_1, W_1) \oplus \Gamma(P_2, W_2) = \Gamma(P, W),$$

if $P = P_1 \cup P_2$ and for any $S \subset P$

$$v(S) = v_1(S \cap P_1) + v_2(S \cap P_2)$$

(where contrary to the composition the addition is meant in Boolean sense).

Similarly, a game is called the product of the same game

$$\Gamma(P_1, W_1) \otimes \Gamma(P_2, W_2) = \Gamma(P, W)$$

if also $P = P_1 \cup P_2$ and for any $S \subset P$

$$v(S) = v_1(S \cap P_1)\, v_2(S \cap P_2)$$

It turns out that the constructions introduced are in some sense invariant with respect to the concept of a solution: The solutions of sums and products can be obtained in a certain way from the solutions of their components.

A step beyond the scope of simple games has been made with a construction by Owen /1/ which has been called the tensor composition of games.

Several classes of simple games have been investigated too. So, already von Neumann and Morgenstern (in Chapter X) have discussed different kinds of majority games.

Bott /1/ has introduced majority (n, k)-games where $n/2<k<n$ assuming

$$v(S) = \begin{cases} -|S| & \text{for} \quad |S|<k \\ n-|S| & \text{for} \quad |S|\geq k, \end{cases}$$

and has found a unique class of solutions passing from one to another for automorphisms of the game. Further results for this class of games have been obtained by Gillies /1/.

8. Along with the solution of a game, a "reasonable" class of imputations is represented by the c-core, examined by Gillies /2/, consisting of all imputations not dominated by any imputation. The "reasonableness" of the c-core is determined by the property that it consists of all imputations $(\alpha_1, \ldots, \alpha_n)$ such that

$$\sum_{i\in S} \alpha_i \geq v(S) \quad ^{*)}$$

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*) Translator's note: In the original Russian text this condition is given in the form of a strict inequality, $\sum_{i\in S} \alpha_i > v(S)$, which is, however, obviously a misprint.

for any coalition S (i.e., if an imputation is in the core, then no coalition is effective for it). Therefore, every coalition is satisfied with an imputation belonging to the c-core and will not make use of its own strategic possibilities.

It is clear that the c-core is a closed, bounded, and convex set contained in any solution of a game. For many games, however, the c-core turns out to be empty. Gillies /2/ has shown that for the existence of a solution of a game coinciding with the c-core, it is sufficient that all values of the characteristic function of a game are smaller than 1/n, where n is the number of players. This result has been subsequently strengthened by O.N. Bondereva /1/, who has begun to systematically use linear programming in the theory of cooperative games.

9. An essentially different approach to cooperative games has been offered by Aumann and Maschler in their remark /1/ which has established the rudiments of a new direction in the theory of cooperative games. This approach is based on an examination in the well-known sense of stable outcomes of a game at which the players arrive as the result of a bargaining process with a perfect exchange of informations, threats, counter threats etc. The set of all the stable outcomes is called the bargaining set of a game and can be determined by solving systems of algebraic linear inequalities. Fundamental facts forming this theory can be found in the paper /2/ by Aumann and Maschler. Further results are contained in papers by Davis and Maschler /1/ and Peleg /1/. Some considerations concerning this trend in game theory can be found in O. Morgenstern's preface to the present book.

10. The researches in the cooperative theory without transferrable utility, but with side payments, have not been numerous. The first attempt of

constructing such a theory on a descriptive level has been undertaken by Shapley and Shubik /1/ (see also § 10.4 of the book by Luce and Raiffa /1/ and the survey article by Aumann /2/).

11. The transition from the classical cooperative theory to the theory of games without side payments results formally in the generalization of the concept of characteristic function (see the survey article by Aumann /2/).

Let I represent the set of players to each of whom corresponds a coordinate of Euclidean space E_I. Each coalition S, a subspace of E_I denoted by E_S, is spanned by the coordinate axes corresponding to the player of S. Points of E_S are called payoff S-vectors.

The fact that the value of the characteristic function for a given coalition S has been equal to v(S) has signified, in the classical theory, the possibility of this coalition forcing an imputation, the sum of whose components, corresponding to the players of S, has not been smaller than v(S). Geometrically this means that the set of guaranteed payoff vectors for coalition S forms a half-space

$$\sum_{i \in S} x_i \leq v(S)$$

Fundamentally, the very existence of side payments allows for distributing in an arbitrary way the entire sum v(S) among the members of the coalition. With abandonment of side payments the picture will change, especially on this point.

Let v(S) be the set of payoff vectors that coalition S can assure itself: This set will be called the value of the characteristic function for coalitions S. It is natural to require that the characteristic function satisfies the following axioms:

1° $v(S)$ is a convex, closed, and non-empty set (convexity reflects the possibility of mixing the strategies, closedness - the natural property of possibilities, and non-emptiness - the fact of participation in the game).

2° If $x \in v(S)$, $y \in E_S$ and $y \leq x$, then $y \in v(S)$ (i.e., the set $v(S)$ possesses only a "northeast" boundary; intuitively this is sufficiently natural: If the coalition is capable of the more, then it is also capable of the less).

3° For coalitions S and T disjoint,

$$v(S) \times v(T) \subset v(S \cup T)$$

(This condition generalizes the classical axiom of superadditivity of the characteristic function: The possibilities of a union are in any case not narrower than the combination of possibilities of coalitions acting separately).

We will now detach a certain part of $v(I)$ adjoining its "northeast" boundary, i.e., for which the following relation holds:

$$v(I) = \{x \in E_I \ / \text{ there esists a vector } y \in H \text{ such that } y \geq x\}$$

H consists of all those payoff vectors which are feasible due to "external" circumstances.

The pair (v,H) is called a <u>game without side payments</u>.

12. According to the analogy to the classical case, the characteristic function serves as a basis for the introduction of the concept of imputation as a payoff vector being at the same time individually and universally rational. In the given case, these conditions of rationality applied to a vector x are accordingly written as

$$x_i \geq \max v(i);$$

there exists no vector $y \in v(I)$ such that $y > x$.

The concept of domination of imputations is transferred almost literally, from the classical case to the case considered: An imputation x dominates an imputation y, via a coalition S, if $x \in v(S)$ and $x_i > y_i$ for all $i \in S$; an imputation x dominates an imputation y if x dominates y via some coalition S.

Domination generates the concept of the solution, the core etc., as well as all the problems connected with them. Since the possibilities of games without side payments are essentially broader than the possibilities of classical games, the construction of many counter examples seems to be easier. Thus, the question concerning the universal solubility of games without side payments, for instance, has been decided in a negative sense earlier than for the classical cooperative games: Stearns (see p.9 of the survey article by Aumann /2/) has shown that there is a 7-person game with no solution.

13. In the construction of a game without side payments from its normal, strategic form, there emerges besides its usual characteristic function (here denoted by v_α), describing those payoff vectors which can be assured by the members of the coalition, a new function describing those payoff vectors the receipt of which cannot be prevented by the other players. This second function also satisfies the axioms of the characteristic function and is denoted by v_β. The functions v_α and v_β are, in a well-known sense, analogous to the maximin and minimax payoffs.

In the classical cooperative theory (i.e., with side payments and transferrable utility), both functions v_α and v_β coincide. In his only, but very substantial paper /1/, Jentzsch has investigated further possibilities for coincidence of the functions v_α and v_β. Games where $v_\alpha = v_\beta$ he has called clear; moreover, he has presented examples of games which are not clear and has asserted that for the clearness of an arbitrary game with a given set of participants, it is necessary that each coalition possess some "social utility function" for side payments. Jentzsch has remarked (but not proved) that the suggested logarithmic Bernoulli scale of individual utility leads to such a social utility function.

14. As in the classical case, the multitude of imputations in a solution (as well as in the core) and of solutions of a game, reduces the normative value of the solution, and there arises the question of selecting for every game, some unique imputation which could be, with sufficient reason, regarded as "fair". For the classical cooperative theory, the Shapley value vector (see II.6.3) has turned out to be such an imputation. In paper /7/ he (Shapley) has succeeded in extending the corresponding definition to the case of games without side payments. Some further results concerning games without side payments as well as a detailed bibliography have been given in Aumann's survey article /2/.

§ 4: Non-coalition and Coalition Games.

1. The classical cooperative theory, as well as the theory of games without side payments deal, fundamentally, with the behavior of a coalition under conditions of its encirclement, reducing most questions to the antagonistic description or at least to antagonistic analogies. It is clear that for a more complete inquiry into game situations, above all, more general principles of

rational behavior of players are necessary.

The decisive step in this direction has been done by Nash /2/ who has extended the basic idea of the maximin principle to arbitrary non-coalition games. His consideration consists in the following.

Let

$$\Gamma = \langle I, \{S_i\}_{i \in I}, \{H_i\}_{i \in I} \rangle$$

be a non-coalition game (where I is the set of players, S_i - the set of all strategies of player i, and H_i - his payoff function). It is natural to assume, as the fundamental guiding principle of the behavior of a player in such a game, the following principle of realizability of the aim: Actions of players are considered to be rational if the situation being the aim of their common efforts is realizable, i.e., if no one of the players is interested in disturbing this situation.

A more formal principle of realizability of the aim looks like the following.

If s is a situation in a game Γ, and s_i is an arbitrary strategy of player i, then by $s||s_i$ is denoted the situation obtained from the situation s as the result of the replacement of player i's strategy in s by his strategy s_i.

The situation s* in a game, is acceptable for player i if for any of his strategies s_i the inequality $H_i(s^*||s_i) \leq H_i(s^*)$ holds*). The situation is in equilibrium if it is acceptable for each player.

*) Translator's note: In the original Russian text the inequality is written conversely ($\geq$) what is obviously due to a misprint.

If one regards the process of playing as the players' selection of strategies on the basis of a preliminary agreement, then in particular, in equilibrium situations and only in them have the players no reasons of their own to break their commitments.

It is not difficult to examine that if the non-coalition game Γ turns out to be antagonistic, then the principle of realizability of the aim changes into the maximin principle, and the equilibrium situations turn out to be saddle points.

2. While formally the equilibrium situations play the same role in the theory of non-coalition games as saddle points do in antagonistic games, their normative meaning is essentially less: The player's knowledge of his strategies that are included in equilibrium situations does not yet assure him the possibility of realizing the optimal way of acting. This is understandable since the non-antagonistic games are, generally speaking, not exhausted by their strategic aspect.

As an example, one can give a non-antagonistic two-person game, well-known under the name "the battle of the sexes". Here every player has two pure strategies and the payoff function is described by the following tables:

Payoff of player 1.

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

Payoff of player 2.

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$$

Obviously, the players' simultaneous selection of their first or second strategies leads to equilibrium situations. Thus, in the given game every pure strategy of a player is in equilibrium (and, by the way, also one of his mixed strategies).

At the same time, the equilibrium situations obtained here are obviously not equal: The first player should evidently prefer the equilibrium situation formed by the first pure strategies and the second player those formed by the second ones*).

Apparently, the choice of one of these equilibrium situations can be decided only after negotiations between the players.

In this example, we hit again on our inability to reduce the cooperative aspect of the problem to the strategic one. The canonical choice among many equilibrium situations of an arbitrary game is a complicated problem. Interesting approaches to these questions are contained in papers by Harsanyi /1/ and also Krelle and Coenen /1/.

3. Fairly general principles of rationality of agreements between the players have been offered by Nash in his paper /2/.

Every agreement between two players leads to their obtaining of some payoffs r_1 and r_2 respectively. This means that every agreement is characterized by a pair of numbers (r_1, r_2) and can be, therefore, represented by a point in the plane. We assume that R is a set of points corresponding to all possible agreements. Among all possible outcomes of agreements one is singled

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*) Translator's note: This is evidently a mistake: The first player will prefer the equilibrium point made up of the second strategies of each player, and the second will prefer the equilibrium point formed by the first strategies of the contestants.

out at which the players arrive if the attempt to come to an agreements fails. In this case, the payoffs of the players are those quantities which the players would obtain by themselves. The point $r^{\circ} = (r_1^{\circ}, r_2^{\circ})$, corresponding to such an outcome, is sometimes called "the status quo".

Let $F(R, r^{\circ})$ be a function defined on the set of all bargaining situations, the values of which are payoff vectors (pairs). This function determines which payoffs have to be considered fair for the players under the conditions of every bargaining situation.

In accordance with Nash, it is natural to require that the fair arbitration scheme fulfils the following axioms:

(1) Effectivity:

$$F(R, r^{\circ}) \geqq r^{\circ}$$

(in other words: The arbitration is fair if no one of the bargaining partners will lose by it).

(2) Symmetry: if the set R is symmetrically located with respect to the bisecant, with an angle of coordinates $r_1 = r_2$, and the components of the vector r° are equal, then the components of the vector $F(R, r^{\circ})$ have to be equal too (players in an equal position in the bargaining situation should obtain equal payoffs).

(3) Pareto optimality: In the arbitration $F(R,r^{\circ})$, both players cannot simultaneously increase their payoffs (this property of the arbitration has nothing to do with its fairness, but with its general rationality: A rational bargain cannot be improved so that each of its participants increases has own payoff).

(4) Monotony with respect to the domain: If $R_1 \subset R_2$ and

$F(R_2, r°) \in R_1$, then $F(R_1, r°) = F(R_2, r°)$ (If for a "large" set of possible bargaining outcomes fairness requires the selection of a bargain in a "smaller" set, then after the transition to the smaller set the old fair agreement remains valid).

(5) Invariance with respect to the choice of the origin and unit of measurement: For any positive k_1 and k_2

$$F\left(\begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} R + r', \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} r° + r'\right) = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix} F(R, r°) + r'$$

(this axiom expresses virtually the linearity of implied utility functions).

The system of axioms formulated is complete. In particular the following theorem holds: If the function $F = (F_1, F_2)$ satisfies the axioms (1) - (5), then

$$(F_1(R, r°) - r_1^°)(F_2(R, r°) - r_2^°) = \max_{\substack{r=(r_1,r_2) \\ \in R}} (r_1 - r_1^°)(r_2 - r_2^°),$$

i.e., a fair bargain is considered to be that one for which the product of the increases in the player's payoffs with respect to the status quo is maximized.

A detailed critical analysis of this sort of questions is contained in the book by Luce and Raiffa /1/.

4. The existence of equilibrium situations in any finite non-coalition game (of course, generally speaking, in mixed strategies) has been proved by Nash /1/. This proof, as well as von Neumann's first proof of the minimax theorem (the natural generalization of which is the former), are based on Brouwer's fixed point theorem. Contrary to the minimax theorem,

however, here one can hardly hope to find a proof making use of sufficiently elementary and "sufficiently effective" procedures (like the theorems of separability of convex sets). The matter is that, as examples have shown, with increasing numbers of players, one has to carry out irrational operations of highest degree in order to find components of equilibrium situations.

Nash's theorem is capable of generalizations for the case of infinite games. Some very interesting examples of theorems of this kind are contained in the book by Burger /1/.

5. In non-coalition games one usually assumes that the players can adopt any probability distributions on the set of pure strategies as their mixed strategies. Wu Wen-tzün /2/ has examined games where the set of pure strategies of every player is given together with some cover, and where solely such mixed strategies are admitted which include only pure strategies that belong to one and the same element of the cover. Such games he has called games with restrictions. In games with restrictions, those situations σ are regarded as equilibrium situations, where $H_i(\sigma \| s_i) \leq H_i(\sigma)$ for any i and all those pure strategies s_i which belong to an element of the cover, the σ_i-probability of which is equal to one. Using very subtle topological arguments, Wu Wen-tzün has ascertained that a sufficient condition for the existence of equilibrium situations in games with restrictions is, first, the coherence of nerves of the covers of the strategy sets of all players and, secondly, that their Euler-Poincaré characteristics are different from zero.

6. No general methods of finding equilibrium situations in non-

coalition games have been known up to now.

An algorithm for the numerical solution for bimatrix games has been given by N. N. Vorob'ev /1/ and improved by Kuhn /2/. A more practical, but not so complete, algorithm is due to Lembke and Howson /1/.

In solving infinite non-coalition games, one has been successful only in some, but not many, cases. A very interesting example of a game of "oligopolists", offered in the book by Burger /1/, belongs to them.

The theory of fair division derived from an ancient problem of dividing an asset according to the principle of "one divides - another chooses", belongs to the class of non-coalition games. From the game-theoretical point of view, Steinhaus (see reports by Knaster and Steinhaus /1/) first dealt with this problem. The latest results in this direction are contained in Kuhn's paper /3/.

7. To every game, there belongs a collection of utilities for which the players are struggling. In the classical cooperative theory (with transferrable utility), there has been only one such utility for all players. In the theory of non-coalition games, there are just as many such utilities as there are players, and each player has his own utility described by his payoff function. A natural and completely topical generalization admitting various interpretations, consists of the correspondence of an (obviously non-empty) set of players to any kind of utility appearing as some players' goals, where, generally speaking, one player may be interested in several utilities. The set of players struggling for the maximization of one and the same utility, is a coalition

(in particular, a coalition of interests). The payoff to every coalition is regarded as belonging to it as such, not being subject to any division among the members of the coalition. It is clear that different coalitions may also intersect each other.

Along with coalitions of interests, one can also meet in games coalitions of actions whose participants are able to exchange information concerning the choice of the common strategy of the coalition and, in particular, to employ correlated randomized actions. The examination of such common randomized actions of intersecting coalitions is complicated by the necessity of coordinating the actions of different groups of players. This may turn out to be a hindrance for the probability interpretation of mixed strategies. Therefore, the "complex of coalitions" cannot be arbitrary and has to be subjected to some combinatorial conditions.

A natural extension the principle of realizability of the aim to coalition games leads to an examination of stable solutions where some groups of players are not interested in deviating from their actions planned even if some of their partners in the coalitions break their original commitments.

In particular, if there is no provision by the rules of the game for a "reaction" by the players to a breach of their partner's obligations, i.e., if the game actually turns into a non-coalition game, then the stable situations described change into equilibrium situations in the sense of Nash (see III.4.1.).

The existence of stable situations of this kind has been proved in N. N. Vorob'ev's paper /5/.

§ 5: Dynamic Games.

1. The utilization of mixed strategies as optimal decisions, the extension of the minimax theorem to various classes of infinite games, the connection of matrix games with linear programming - all this has overshadowed what maybe even deeper ideas contained in the monograph by von Neumann and Morgenstern: The theory of extensive games and the cooperative theory. It suffices to say that until the publication of a volume of collected papers /3/ in 1953, no one had returned to these questions. (Moreover, in Wald's book "Statistical decision functions" from 1950, the game-theoretical side of the problem has been presented by games in normal form in spite of the whole essential "extensiveness" of the problem.)

We note now, that after the publication of von Neumann's and Morgenstern's monograph, these questions have been found in a completely different stage. The cooperative theory has been elaborated very deeply: Its results have concerned the central questions of the theory and some particular cases have been analysed in exhaustive detail. The theory of extensive games has been limited to a cumbersome system of definitions and the theorem on games with perfect information (well-known, moreover, in its basic features from the time of Zermelo's /1/ paper on chess).

2. Of so much greater interest is Kuhn's paper /1/. Here, above all, a natural and transparent definition of the extensive game has been formulated including the precisely defined concept of strategies of the players as functions on the sets of their information states (intuitively, - on the families of sets of positions indistinguishable for the player

at the corresponding instant; these sets of positions are called information sets). Besides that, in this paper the "inner" structure of that uncertainty which the player has to cope with in the extensive game has been elucidated. As such, as an "atom" of uncertainty Kuhn has chosen the fact of "overlaying" of an information set by another one. Intuitively, the "overlaying" of an information set U by an information set V, means that a player in the set V*) does not know whether the game has earlier taken place in the set U and what decision has been made in U (apparently by that player who was to move in U). Obviously, in games with perfect information where every information set consists of a unique position, no information sets are overlaying others.

3. Generally speaking, the "more" overlayings there are among the information sets in a game, the "greater" the necessary mixing in optimal (or if we have to deal with general non-coalition games, in equilibrium) strategies. Therefore, if the player possesses perfect recall, i.e., if his own information sets do not overlay each other, then the player can confine himself to behavioral strategies, i.e., to such mixed strategies where his actions are mixed independently in different information sets. A further development of these ideas is contained in the paper by Thompson /1/ and by N. N. Vorob'ev /2,6/.

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*) Translator's note: In the original Russian text this information set is denoted by U, which is, however, obviously a misprint.

In every extensive game, the set of positions together with the description of the possible transitions from one position to another, i.e., a subject studied by the theory of graphs, plays an essential role. A systematical examination of the combinatorial aspects of extensive games requires the application of terminology and methods of the theory of graphs. An extremely great number of game-theoretical results have been exposed from this position by Berge in his book /1/.

The partition of information sets into smaller information sets, intuitively means an increase of the players' information and formally - an expansion of the sets of their strategies. In particular, after the complete partitioning of information sets into one-position sets, the extended set of strategies obtained will contain optimal (correspondingly equilibrium) strategies. It is clear that a complete partitioning is sufficient for the existence of pure optimal strategies, but not necessary. Necessary conditions for such a partitioning have been given by Birch /1/.

In the theory of extensive games stemming from Kuhn's paper /1/, it has been assumed that information sets must not "precede themselves", i.e., a player must not make decisions twice in one and the same information state in a continuing play. This restriction is not quite natural and therefore somewhat restraining. Isbell /1/ has initiated the theory of "finitary" games not restricted by this condition.

4. At first sight the theorem of the existence of equilibrium situations in pure strategies for games with perfect information seems to be a very general fact capable of broad generalizations. Gale and Stewart /1/,

however, have given an example of an extensive game (of course, an infinite one) with perfect information where the players do not have pure optimal strategies. This example has shown that the theory of infinite extensive games with perfect information is extremely complicated. Further results in this direction are due to Wolfe /1/, Oxtoby /1/, Hanani /1/, Stocki /1/, Davis /1/, Mycielski /1/ and others.

At the same time, a very straightforward utilization of the axiom of choice at the construction of this example has given reason for assuming that it belongs to the category of paradoxes generated by the application of this axiom. Steinhaus and Mycielski /1/ have occupied themselves with the consistent development of such a point of view. They have assumed the strict determinateness of an antagonistic game with perfect information as an independent axiom of set theory (in its abstract formulation, obviously, one can do this without using game-theoretical terminology), and have named it the <u>axiom of determinateness</u>. The intuitive meaning of this axiom consists in the fact that for participants who are absolutely informed about the rules and the details of the game, the outcome of the play is predetermined. Such a game has to be a purely combinatorial one. It is clear that the axiom of determinateness is inconsistent with Gale's and Stewart's example. In fact, it is inconsistent with the axiom of choice applied in the construction of this example. To a great extent, the axiom of determinateness is also apt to replace the axiom of choice. Mycielski /2/ has shown that one succeeds in proving a great number of consequences of the axiom of choice on the basis of the axiom of determinateness, without using the axiom of the choice. In particular, in his paper written with Swierczkowski /1/, the

Lebesque measurability of any linear space has been derived from the axiom of determinateness. A further study of games with perfect information has been carried out by Mycielski in his paper /1/.

5. In extensive games, payoffs are received by the players only at end positions of the game. Therefore, formally speaking, the players can compare among themselves only end positions. The player's occupation of any position, however, (speaking for the sake of security about antagonistic games) already predetermines a certain payoff by which it is possible to measure the value of the position itself. Thus, the transition of the play from position to position can mean for the player a gaining of some temporary advantages which may get lost completely in a further non-optimal play. In connection with all that has been said above, it seems expedient to examine such games where the immediate struggle is carried on for some positions of a game which turn out to be specific recources of the players in the course of their further struggle. Such an approach applied to chess has been developed in detail by M. M. Botvinnik /1/.

In the literature, very many games of this kind have been examined. One of the most general is the scheme of the recursive game introduced by Everett /1/.

The play of a recursive game consists of a sequence of elementary (for instance, matrix) games where the outcome of every elementary game is again an elementary game or the end of the whole game accompanied by some payoff. Speaking more precisely, if at some moment of time an elementary game Γ^k was played where the players chose the strategies i, j, then with certain probabilities (depending on the strategies chosen) at the next moment either some elementary game Γ^1 will be played from the same set, or the whole play will terminate. It turns out that if the elementary games possess values, then recursive games have values in stationary strategies too, and therefore ε-optimal stationary strategies for any $\varepsilon>0$.

6. Recursive games are closely related to games of survival investigated by Milnor and Shapley /1/. In such games, at any instant of time the players have r and R - r $(0<r<R)$ resources respectively, and play a matrix game $||a_{ij}||$. The payoffs in this game are added to the player's resources with which they enter into the game at the next instant. The game terminates and the winner obtains one unit (whose dimension, generally speaking, is not connected with the dimension of resources) when the resources of one of the players are exhausted.

If such a game has a value, then it is a function of the initial amount of player 1's resources r_o, which is a monotonic solution of the functional equation

$$\varphi(r) = \text{val} \quad ||\varphi(r + a_{ij})||$$

with boundary conditions

$$\varphi(r) = \begin{matrix} 0 & \text{if } r\leq 0 \\ 1 & \text{if } r\geq R. \end{matrix}$$

A generalization of this game can be reached by passing from matrix games to arbitrary non-coalition games, or, what is virtually the same thing, to vector resources. Games of this kind have been examined by I. V. Romanovski /1/.

Attrition games are similar to games of survival; as to them, see the paper by Isbell and Marlow /1/.

7. New difficulties of fundamental character emerge from passing to the study of dynamic games where the players' decisions are made not at discrete instants, but are a process continuous in time. The fundamental class of such games studied is the class of differential games.

A differential game can be schematically described in the following way. Let there be given a bounded, connected open subset A of a finitely dimensional Euclidean space, the starting point x° and the set of differential equations

$$\frac{dx_j}{dt} = f_j(x_1, \ldots, x_n; t, \varphi, \psi), \; j = 1, \ldots, n. \qquad (5)$$

The variables x_j are called state variables (or phase coordinates), φ and ψ - control variables (for the first and second player respectively) which are selected by the two players from suitably prescribed sets of functions depending on the time parameter t and on the state variables. If the selection of functions φ and ψ results in the solvability of the system of differential equations, then the play of the game is represented by a trajectory in the set A. The payoff function is defined on the set of all trajectories. It is usually assumed that the game terminates when the trajectory attains the boundary of A, and the payoffs are determined at the boundary points. If a trajectory stays within A for an infinite time, then some payoff is defined for it too.

In the definition quoted, some inaccuracy in the definition of the strategy sets of the players attracts attention; For some functions φ and ψ being strategies, it is obviously necessary that inserting these φ and ψ into (5) yields a system of differential equations having a unique solution.

A typical example for a differential game is the game of "pursuit" where the phase coordinates determine the position (sometimes the speed too) of certain objects, some of which are called pursuers and others evaders. Every player controls his objects through the coordinates. The game terminates at a moment fixed in advance where the pursuers' payoff is determined by the "nearness" of the pursuing objects to the evading ones at the moment of termination of the game. (The first mathematical description of the game of pursuit is due to Warmus /1/.)

A second example is provided by the game of "pulling over", where two players exert forces on some material point striving to attach the desired phase coordinates to it at the end of the game. Obviously, pursuit games can be regarded as a subcase of the games of "pulling over" (in the corresponding phase space).

A systematic examination of certain examples of games of this kind, as well as of general theoretical considerations, has been undertaken by Isaacs in the early 'fifties. His first papers appeared as Rand Corporation memoranda in 1954 and 1955, and a detailed exposition of the results obtained has been published in the form of a monograph /1/. In fact, Isaacs has obtained his results by applying Bellman's method of dynamic programming and therefore we are meeting all the difficulties due to this method. Further results in this direction have been obtained by L. A. Petrosian /1,2/. A more subtle and powerful approach can be found in L. S. Pontryagin's papers /1/. This approach is based on reasonings closely related to those which lead to the maximum principle. Here the solution of a differential game is reduced to the solution of the system of ordinary differential equations.

Attempts have been made to reduce the solution of a differential game to the solution of a game with discrete time and a subsequent limit operation. Here papers by Scarf /1/ and Fleming /1,2,3/ are noteworthy.

Thorough investigations of differential games based on the utilization of the apparatus and results of the calculus of variations have been carried out by Fleming and Berkovitz /1/.

A very detailed survey on contributions to the theory of differential games is contained in papers by Simakova /1/ and also Simakova and Zelikin /1/.

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